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EIGENFUNCTION EXPANSIONS FOR NON-SYMMETRIC PARTIAL DIFFERENTIAL OPERATORS, II.*

By Felix E. Browder.[1]

Introduction. In a preceding paper [7], we have established an eigenfunction expansion theorem of the Weyl-Plancherel type for any pair of partial differential operators L and B having a realization A which is a subnormal operator in the Hilbert space H_B defined by the positive operator B.[2] It is the purpose of the present paper to extend this result to a much more general class of pairs (L,B) by interpreting the notion of eigenfunction expansion in a more general sense. The class of realization operators for which our results are valid include a significantly large subclass of the unbounded spectral operators in the sense of Dunford ([1], [9]). In addition, the proofs of the expansion theorems of the present paper, which are of a different type than those of [7], enable us to overcome the technical difficulties which arose in [7] as to the nature of the eigenfunctions obtained when the order of B is different from zero.

Let us state the results to be obtained in a more precise way. Let L and B be two partial differential operators defined on an open set G of the n-dimensional Euclidean space E^n, L' the adjoint differential operator of L, $C_c^\infty(G)$ the family of infinitely differentiable functions with compact support in G, and (u,v) the inner product in $L^2(G)$.

We suppose that B is positive, i. e. that $(Bu,u) > 0$ for each u in $C_c^\infty(G)$, and that the Hilbert space H_B with inner product $[\ ,\ ]$ obtained by completing $C_c^\infty(G)$ with respect to the norm $\| u \|_B = (Bu,u)^{\frac{1}{2}}$, has a continuous imbedding in the space of distributions on G. The minimal realization of the pair (L,B) in H_B is the operator A_0 in H_B with domain $C_c^\infty(G)$ defined uniquely by the condition $[A_0u,v] = (Lu,v), u,v \in C_c^\infty(G)$.

If, by a similar definition, the operator A'_0 in H_B is the minimal realization of the pair (L',B) in H_B, we define the maximal realization of (L,B)

* Received April 1, 1958.

[1] This paper was written while the writer held a National Science Foundation Senior Post-Doctoral Fellowship.

[2] A summary of recent literature on the general theory of eigenfunction expansions for partial differential operators (including [2], [3], [4], [5], [10], [11], [12], [13], [14], and [18]) is given in the Introduction to [7].

to be $A_1 = (A'_0)^*$. More generally, an operator A in H_B will be said to be a realization of the pair (L, B) if $A_0 \subseteq A \subseteq A_1$.

A distribution u on G is said to be an eigenfunction of order one of the pair (L, B) with eigenvalue ζ (ζ a complex number) if $(L - \zeta B)u = 0$. For $k > 1$, by recursion, we define u to be an eigenfunction of order k of the pair (L, B) with eigenvalue ζ if $(L - \zeta B)u = Bv$, where v is an eigenfunction of order $(k-1)$ of (L, B) with eigenvalue ζ. (In an intuitive sense, eigenfunctions of order k correspond to generalized solutions of the equation $(A_1 - \zeta I)^k u = 0$.)

Let C^1 be the set of complex numbers, Ω the σ-algebra of Borel subsets of C^1. In terms of the generalized definition of eigenfunction, the notion of eigenfunction expansion can be described precisely as follows:

Definition 1. An eigenfunction expansion for (L, B) consists of a finite measure m on Ω and two double sequences $\{e_{jk}\}$ and $\{f_{jk}\}$ of functions from C^1 to the space of distributions on G such that:

(a) For all j, k, and ζ, $e_{jk}(\zeta)$ is an eigenfunction of order k of the pair (L, B) with eigenvalue ζ.

(b) For each u in $C_c^\infty(G)$, the function $c_{jk}(\zeta) = (Bu, f_{jk}(\zeta))$ lies in $L^2(m)$, and the mapping U_{jk} defined by $U_{jk}(u) = c_{jk}$ can be extended by continuity to a bounded linear mapping of H_B into $L^2(m)$.

(c) For each c in $L^2(m)$, the integral $\int_{C^1} e_{jk}(\zeta) c(\zeta)\, dm(\zeta)$ converges in the distribution topology to an element h_{jk} of H_B, and the mapping V_{jk} of $L^2(m)$ into H_B defined by $V_{jk}c = h_{jk}$ is a bounded linear mapping.

(d) For each u in H_B, $u = \sum_{j,k} V_{jk} U_{jk} u$.

We remark, writing property (d) out formally, that

$$u(x) = \sum_{j,k} \int_{C^1} e_{jk}(\zeta) c_{jk}(\zeta)\, dm(\zeta),$$

justifying the terminology of 'eigenfunction expansion.'

Definition 2. The eigenfunction expansion of Definition 1 is said to be regular if, for each ζ in C^1, $f_{jk}(\zeta)$ is an eigenfunction of the pair (L', B) with eigenvalue $\bar{\zeta}$.

Definition 3. The eigenfunction expansion of Definition 1 is said to be normal if $e_{jk}(\zeta) = f_{jk}(\zeta)$ for all j, k, and ζ, while $V_{jk} = U_{jk}^*$.

We shall prove in Section 4 that an eigenfunction expansion in the sense

of Definition 1 exists for each pair (L, B) having a realization A which belongs to the class of decomposable operators defined below, a regular expansion if A is decomposable of finite order, and a normal expansion if A has a normal decomposition.

The operator A is decomposable if it possesses a decomposition, which, roughly speaking, is a certain infinite-dimensional analogue of the triangular form for finite matrices.[3] More precisely, a decomposition for A consists of the following: a finite measure m on Ω, a bounded linear mapping U of H_B into $L = \sum_{j,k} \oplus (L^2(m))_{jk}$, a bounded linear mapping V of L into H_B, and two bounded linear transformations F and G of L of the form

$$F(\{\alpha_{rs}\})_{jk} = \textstyle\sum_{s>k} \sum_r f_{jkrs}\alpha_{rs},$$

$$G(\{\alpha_{rs}\})_{jk} = \textstyle\sum_{s>k} \sum_r g_{rsjk}\alpha_{rs},$$

such that:

(a) $VU = I$.

(b) $U(Au) = M_\zeta Uu + FUu,\ u \in C_c^\infty(G)$ (M_ζ is the multiplication operator by ζ in each component of L).

(c)′ For $\{\alpha_{jk}\}$ in $L^2(m)$ with uniformly bounded support, $\{\alpha_{jk}\} \in L$, $V\{\alpha_{jk}\}$ lies in $D(A)$, and

$$AV(\{\alpha_{rs}\}) = VM_\zeta(\{\alpha_{rs}\}) + VG(\{\alpha_{rs}\}).$$

The decomposition is of order less than k_0 if $U_{jk} = V_{jk} = 0$ for $k \geqq k_0$, and normal if $V = U^*$.

Every operator of the form $A = S + N$, where S is an unbounded scalar operator in the sense of Dunford and N is nilpotent and commutes with the spectral measure of S, is decomposable of finite order. If $A = S + N$, with S a scalar operator, N commuting with the spectral measure of S, and $N^{j(u)}u = 0$ for u in a dense subset of H_B, then A is decomposable.

A is said to be weakly normally decomposable if only (a) and (b) hold while $V_{jk} = U_{jk}^*$. Every restriction of a normally decomposable operator is weakly normally decomposable, so that, in particular, subnormal operators fall in this class. For weakly normally decomposable operators of finite order, we also obtain an eigenfunction expansion theorem.

For subnormal operators, our present results are a sharpening of those of [7] since all the eigenfunctions obtained are distributions rather than

[3] This analogue of the triangular form has very little in common with the generalized triangular form constructed by Lifschitz [15] for operators whose imaginary part has a finite trace.

linear functionals of a more general type. The improvement is brought about by replacing the Banach space differentiation theorem of Birkhoff-Gelfand used in [7] by a result established below on the representation of general linear transformations from an L^p space into the space of distributions of G. For hypoelliptic pairs (L, B), i.e. those for which the distribution solutions of $(L - \zeta B)u = 0$ are differentiable functions for every ζ, we obtain as a result an eigenfunction expansion theorem with smooth eigenfunctions.

Section 1 is devoted to the theory of decomposable and weakly decomposable operators in a separable Hilbert space. Section 2 discusses the functional properties of the space H_B. Section 3 gives the proof of the representation theorem for continuous linear mappings of $L^p(m)$ into $\mathfrak{D}'(G)$. Section 4 contains the proof of the expansion theorems for decomposable and weakly decomposable operators. Section 5 gives the application of the expansion theorem to hypoelliptic pairs.

1. Let H be a separable Hilbert space with inner product $[u, v]$, S_0 a dense subset of H. Let T be a linear operator in H with domain $D(T)$ dense in H and range $R(T)$. We suppose that T^*, the adjoint of T, is densely defined in H and that $S_0 \subset D(T) \cap D(T^*)$.

Definition 1.1. Let m be a finite measure on Ω, $\{f_{jkrs}\}$ and $\{g_{jkrs}\}$ two quadruple sequences of m-essentially bounded functions on C^1. Suppose further we are given $\{U_{jk}\}$ a double sequence of bounded linear mappings of H into $L^2(m)$, $\{V_{jk}\}$ a double sequence of bounded linear mappings of $L^2(m)$ into H. Let $L = \sum_{j,k} \oplus (L^2(m))_{jk}$. If $\{\alpha_{rs}\}$ is an element of L, we define the mappings F and G of L into itself by

$$F(\{\alpha_{rs}\})_{jk} = \sum_{s>k} \sum_r f_{jkrs}\alpha_{rs},$$

$$G(\{\alpha_{rs}\})_{jk} = \sum_{s>k} \sum_r g_{rsjk}\alpha_{rs}.$$

In addition, the mappings U of H into L, V of L into H are defined formally by

$$(Uu)_{jk} = U_{jk}u, \qquad V(\{\alpha_{jk}\}) = \sum_{jk} V_{jk}(\alpha_{jk}).$$

The family $(m, \{f_{jkrs}\}, \{g_{jkrs}\}, \{U_{jk}\}, \{V_{jk}\})$ is said to be a permissible system if the mappings F, G, U, and V defined above are all bounded linear mappings.

Definition 1.2. The permissible system $(m, \{f_{jkrs}\}, \{g_{jkrs}\}, \{U_{jk}\}, \{V_{jk}\})$ is said to define a decomposition for the operator T in H with respect to the subset S_0 of H if:

(a) For every u in H, $u = \sum_{jk} V_{jk} U_{jk} u$.

(b) For u in S_0 and ζ in the complement of an m-null set,

$$U_{jk}(Tu)(\zeta) = \zeta U_{jk}(u)(\zeta) + \sum_{s>k} \sum_r f_{jkrs}(\zeta) U_{rs}(u)(\zeta).$$

(c) For u in S_0, α in $L^2(m)$ with bounded support in C^1,

$$[V_{jk}(\alpha), T^* u] = [V_{jk}(\zeta\alpha), u] + \sum_{s>k} \sum_r [V_{rs}(g_{jkrs}\alpha), u].$$

We note that the sum in the last term in the right-hand side of (b) is precisely $(FUu)_{jk}$ as defined in Definition (1.1) and is, therefore, a well-defined element of $L^2(m)$. Similarly, the sum which is the last term in the right-hand side of (c) is equal to $[h, u]$, where $h = VGB_{jk}(\alpha)$, B_{jk} being the injection mapping of $L^2(m)$ into the (j,k)-th component of L.

The following property, which we shall have occasion to use in the discussion of this Section, implies property (c) of Definition (1.2) for any subset S_0 of $D(T^*)$:

(c′) For α in $L^2(m)$ with bounded support in C^1, $V_{jk}(\alpha)$ lies in $D(T)$ and

$$TV_{jk}(\alpha) = V_{jk}(\zeta\alpha) + \sum_{s<k} \sum_r V_{rs}(g_{jkrs}\alpha).$$

Definition 1.3. The decomposition of Definition 1.2 is said to be normal if $V_{jk} = U_{jk}^*$.

If a decomposition is normal, the mapping U of H into L as given in Definition (1.1) is an isometric mapping.

THEOREM 1.1. *Let T be an operator in H such that $T = RT_1R^{-1}$, where T_1 has a decomposition with respect to S_0 satisfying* (c′) *and R is a bicontinuous linear mapping of H onto H. Then T has a decomposition with respect to $R(S_0)$ satisfying* (c′).

Proof. Let $(m, \{f_{jkrs}\}, \{g_{jkrs}\}, \{U^1{}_{jk}\}, \{V^1{}_{jk}\})$ be the permissible system defining the decomposition for T_1 with respect to S of the hypothesis. We define $U_{jk} = U^1{}_{jk}R^{-1}$, $V_{jk} = RV^1{}_{jk}$. It follows easily that $(m, \{f_{jkrs}\}, \{g_{jkrs}\}, \{U_{jk}\}, \{V_{jk}\})$ is a permissible system. We shall verify by direct computation that properties (a) and (b) of Definition (1.2) are satisfied for this system with respect to T and $R(S_0)$, as well as property (c′) above.

For (a): $\sum_{jk} V_{jk} U_{jk} u = R(\sum_{jk} V^1{}_{jk} U^1{}_{jk})(R^{-1}u) = RR^{-1}u = u$, for all u in H.

For (b): If $u \in R(S_0)$, then $R^{-1}u \in S_0$, and we have

$$\begin{aligned} U_{jk}(Tu)(\zeta) &= U^1{}_{jk}R^{-1}(RT_1R^{-1}u)(\zeta) = U^1{}_{jk}(T_1R^{-1}u)(\zeta) \\ &= \zeta U^1{}_{jk}(R^{-1}u)(\zeta) + \textstyle\sum_{s>k}\sum_r f_{jkrs}(\zeta)U^1{}_{rs}(R^{-1}u)(\zeta) \\ &= \zeta U_{jk}(u)(\zeta) + \textstyle\sum_{s>k}\sum_r f_{jkrs}(\zeta)U_{rs}(u)(\zeta). \end{aligned}$$

For (c'): $T(V_{jk}(\alpha)) = RT_1R^{-1}RV^1{}_{jk}(\alpha) = RT_1V^1{}_{jk}(\alpha)$

$$= RV^1{}_{jk}(\zeta\alpha) + \textstyle\sum_{s<k}\sum_r RV^1{}_{rs}(g_{jkrs}\alpha) = V_{jk}(\zeta\alpha) + \sum_{s<k}\sum_r V_{rs}(g_{jkrs}\alpha).$$

Definition 1.4. The permissible system $(m, \{f_{jkrs}\}, \{U_{jk}\}, \{V_{jk}\})$ is said to define a weak normal decomposition for T with respect to S_0 if (a) and (b) of Definition (1.2) hold while $V_{jk} = U_{jk}{}^*$.

THEOREM 1.2. *Let T be an operator in the Hilbert space H, which is a closed subspace of the Hilbert space H_1. Suppose that $T \subseteq T_1$, where T_1 is a normally decomposable operator in H_1 with respect to S_1, $S_0 \subset S_1$. Then T is weakly normally decomposable in H with respect to S_0.*

Proof. Let $(m, \{f_{jkrs}\}, \{g_{jkrs}\}, \{U^1{}_{jk}\}, \{V^1{}_{jk}\})$ be a permissible system defining a normal decomposition for T_1 in H_1 with respect to S_1. Let U_{jk} be the restriction of $U^1{}_{jk}$ to H, and $V_{jk} = PV^1{}_{jk}$, where P is the projection mapping of H_1 on H. We shall verify that $(m, \{f_{jkrs}\}, \{U_{jk}\}, \{V_{jk}\})$, which is obviously a permissible system, defines a weak normal decomposition for T in H with respect to S_0.

For (a): $\sum_{jk} V_{jk}U_{jk}u = P(\sum_{jk} V^1{}_{jk}U^1{}_{jk}u) = Pu = u$, for u in H.

For (b): This follows directly from the fact that $T \subseteq T_1$ and that $U_{jk}u = U^1{}_{jk}u$ for u in H.

To prove $V_{jk} = U_{jk}{}^*$: We remark by hypothesis, $V^1{}_{jk} = (U^1{}_{jk})^*$. Since U_{jk} is the restriction of $U^1{}_{jk}$ to H, its adjoint is the projection of the adjoint of $U^1{}_{jk}$ into H.

Following the terminology of the theory of spectral operators ([9], [1]), we shall denote by a spectral measure on H, a function E from Ω to the bounded operators on H such that:

(i) $E(\phi) = 0$ (ϕ the null set), $E(C^1) = I$.

(ii) For all σ_1, σ_2 in Ω,

$$E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2),$$
$$E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2) - E(\sigma_1)E(\sigma_2).$$

(iii) There exists a constant M with $\| E(\sigma) \| \leqq M$ for all σ in Ω.

(iv) For each x and y in H, $[E(\sigma)x, y]$ is countably additive on Ω.

The scalar operator S associated with the spectral measure E is defined by

$$[Su, v] = \int \zeta \, d[E_\zeta u, v]$$

for all u in H for which the integral converges for all v in H.

By a theorem due to Mackey [17] and Lorch [16] (a complete exposition of the proof is given in [20]), for each spectral measure E in H, there exists a bicontinuous linear operator R mapping H onto itself such that $R^{-1}E(\sigma)R$ is an orthogonal projection for every σ in Ω. It follows directly that S is closed and densely defined, that $R^{-1}SR$ is a normal operator in H, and that in (iv) above, we may replace weak countable additivity by strong.

The class of scalar operators in a Hilbert space might thus be defined as the smallest class of (possibly) unbounded operators containing the normal operators which, with each S, contains all operators of the form $R^{-1}SR$, for all bicontinuous linear mappings R of H onto H.

Let N be a bounded operator in H. N is said to commute with the spectral measure E if $E(\sigma)N = NE(\sigma)$ for every σ in Ω. N is said to be semi-nilpotent if there is a dense subset D_1 of H such that for u in D_1, there exists an integer $j(u)$ with $N^{j(u)}u = 0$. By a simple category argument, it follows that if $D_1 = H$, then N is actually nilpotent.

The closed operator T is said to be a spectral operator with respect to the spectral measure E provided that: (1) $E(\sigma)u \in D(T)$ for each u in H and every bounded Borel set σ; (2) For each σ in Ω, $E(\sigma)D(T) \subset D(T)$, while $E(\sigma)Tu = TE(\sigma)u$ for u in $D(T)$; (3) The spectrum of the operator T restricted to the space $E(\sigma)H$ is contained in the closure of σ. If T is bounded, Dunford [9] has shown that $T = S + N$, where S is the scalar operator associated with E and N is a generalized nilpotent ($\| N^n \|^{1/n} \to 0$ as $n \to \infty$) which commutes with the spectral measure E. Conversely, each such operator is a spectral operator. If S is unbounded, N a bounded operator commuting with the spectral measure E such that N is a generalized nilpotent on $E(\sigma)H$ for every bounded Borel set σ, Bade [1] has shown that $S + N$ is a spectral operator.

We shall consider operators of the form $S + N$, where S is a scalar operator (possibly unbounded), N a bounded semi-nilpotent operator commuting with the spectral measure E of S. It follows from the result of Bade that if N is nilpotent, our operators are spectral operators.

THEOREM 1.3. *Let S be a scalar operator with respect to the spectral measure E on the separable Hilbert space H, N a semi-nilpotent operator commuting with E. Then $T = S + N$ is a decomposable operator with a*

decomposition satisfying property (c′). *If* N *is nilpotent, the decomposition is of finite order. If* $N=0$, *the decomposition may be taken of first order.*

Proof. For the subset S_0 with respect to which the decomposition is defined, we choose $D(T)\cap D(T^*)$. It follows by the application of Theorem (1.1) that it suffices to consider the case when S is normal, $E(\sigma)$ a family of orthogonal projections forming the spectral family of S in the sense of the classical spectral theorem for normal operators.

Let $H_k=\{u \mid N^k u=0\}$. H_k is a closed subspace of H for each k, and $D_1=\bigcup_k H_k$. Since $E(\sigma)N^k u=N^k E(\sigma)u$ for every σ, it follows that $E(\sigma)H_k\subset H_k$. For $g\in H$, let $H(g)$ be the cyclic subspace generated by g with respect to the spectral measure E, i.e. the span in H of the family $\{E(\sigma)g\colon \sigma\in\Omega\}$. Since H_k is closed, if $g\in H_k$, $H(g)\subset H_k$.

Let $H_k'=H_k\ominus H_{k-1}$. Since $E(\sigma)H_k\subset H_k$ for every k while $E(\sigma)$ is self-adjoint, it follows that $E(\sigma)H_k'\subset H_k'$. Thus if $g\in H_k'$, $H(g)\subset H_k'$. Since H and therefore H_k' are separable Hilbert spaces, we may choose a sequence $\{g_{jk}\colon j=1,2,\cdots\}$ in H_k' such that $H_k'=\sum_j\oplus H(g_{jk})$. We observe that $H=\sum_{j,k}\oplus H(g_{jk})$, since the dense subset D_1 is contained in the right-hand side. We choose the g_{jk} such that $\|g_{jk}\|=1$.

Let $m_{jk}(\sigma)=[E(\sigma)g_{jk},g_{jk}]$, $m(\sigma)=\sum_{jk}2^{-j-k}m_{jk}(\sigma)$. Let ψ_{jk} be the Radon-Nikodym derivative of m_{jk} with respect to m. Let R_{jk} be the isometric mapping of $L^2(m_{jk})$ into $L^2(m)$ defined by $R_{jk}\alpha=\psi^{\frac{1}{2}}\alpha$, K_{jk} the projection of $L^2(m)$ onto the image of $R_{jk}\cdot K_{jk}\alpha=\Xi_{jk}\alpha$, where Ξ_{jk} is the characteristic function of the set where $\psi_{jk}\neq 0$.

Let W_{jk} be the mapping of $L^2(m_{jk})$ into $H(g_{jk})$ defined in the usual sense by

$$W_{jk}\alpha=\int_{C^1}\alpha(\zeta)\,dE_\zeta g_{jk},\qquad \alpha\in L^2(m_{jk}).$$

Then W_{jk} is a unitary mapping of $L^2(m_{jk})$ onto $H(g_{jk})$, whose inverse we denote by J_{jk}. We denote by P_{jk} the projection mapping of H on $H(g_{jk})$.

We define the mappings U_{jk} of H into $L^2(m)$, V_{jk} of $L^2(m)$ into H by

$$U_{jk}u=R_{jk}J_{jk}P_{jk}u,$$
$$V_{jk}\alpha=W_{jk}R_{jk}^{-1}K_{jk}\alpha.$$

It follows immediately that $V_{jk}=(U_{jk})^*$.

We note that $N(H_k')\subset H_{k-1}=\sum_{s<k}\sum_r\oplus H(g_{rs})$. Thus for each j and k, $Ng_{jk}=\sum_{s<k}\sum_r W_{rs}(h_{jkrs})$, with h_{jkrs} an uniquely defined element of $L^2(m_{jk})$. We define the quadruple sequences $\{f_{jkrs}\}$ and $\{g_{jkrs}\}$ by

$$g_{jkrs}=\Xi_{jk}\psi_{jk}^{-\frac{1}{2}}\psi_{rs}^{\frac{1}{2}}h_{jkrs},\ f_{jkrs}=g_{rsjk}.$$

Let α be a bounded Borel-measurable function with bounded support in C^1 such that there exists $\xi > 0$ for which the support of α is contained in the set $\{\zeta: \psi_{jk}(\zeta) = 0, \text{ or } \psi_{jk}(\zeta) \geqq \xi\}$. (The set of such α is dense in $L^2(m)$, by a standard argument.) From the definition of the mapping V_{jk},

$$V_{jk}(\alpha) = \int_{C^1} \alpha(\zeta)\Xi_{jk}(\zeta)\psi_{jk}(\zeta)^{-\frac{1}{2}}dE_\zeta g_{jk}.$$

Since N is a bounded mapping and commutes with the spectral family $\{E_\zeta\}$,

$$\begin{aligned} NV_{jk}(\alpha) &= \int_{C^1} \alpha(\zeta)\Xi_{jk}(\zeta)\psi_{jk}(\zeta)^{-\frac{1}{2}}\, dE_\zeta(Ng_{jk}) \\ &= \textstyle\sum_{s<k}\sum_r \displaystyle\int_{C^1} \alpha(\zeta)\Xi_{jk}(\zeta)\psi_{jk}(\zeta)^{-\frac{1}{2}}h_{jkrs}(\zeta)\, dE_\zeta g_{rs} \\ &= \textstyle\sum_{s<k}\sum_r V_{rs}(\Xi_{jk}\psi_{jk}^{-\frac{1}{2}}\psi_{rs}^{\frac{1}{2}}h_{jkrs}\alpha) = \sum_{s<k}\sum_r V_{rs}(f_{jkrs}\alpha). \end{aligned}$$

It follows by standard arguments that the functions g_{jkrs} are all m-essentially bounded, that $\sum_{s<k}\sum_r m\text{-ess.}\sup|g_{jkrs}|^2 < \infty$, the equality $NV_{jk}(\alpha) = \sum_{s<k}\sum_r V_{rs}(g_{jkrs}\alpha)$ holds for all α in $L^2(m)$, and, since $T = S + N$, for α in $L^2(m)$ with bounded support,

$$TV_{jk}(\alpha) = V_{jk}(\zeta\alpha) + \textstyle\sum_{s<k}\sum_r V_{rs}(g_{jkrs}\alpha).$$

If α and β in $L^2(m)$ have bounded supports, then,

$$\begin{aligned} &[N^*V_{jk}\alpha, V_{rs}\beta][V_{jk}\alpha, NV_{rs}\beta] = \textstyle\sum_{t<s}\sum_q[V_{jk}\alpha, V_{qt}(g_{rsqt}\beta)] \\ &\quad = [V_{jk}\alpha, V_{jk}(g_{rsjk}\beta)] = \displaystyle\int_{C^1} \alpha(\zeta)\bar{g}_{rsjk}(\zeta)\beta(\zeta)\,dm(\zeta) \\ &\quad = [V_{rs}(\bar{f}_{jkrs}\alpha), V_{rs}\beta], \end{aligned}$$

for $k < s$, while for $k \geqq s$, $[N^*V_{jk}\alpha, V_{rs}\beta] = 0$ by the same computation. It follows that

$$N^*V_{jk}\alpha = \textstyle\sum_{s>k}\sum_r V_{rs}(\bar{f}_{jkrs}\alpha).$$

Let $u \in D(T)$, α in $L^2(m)$ with bounded support. Then,

$$\begin{aligned} &\int_{C^1} U_{jk}(Tu)(\zeta)\bar{\alpha}(\zeta)\,dm(\zeta) = [Tu, V_{jk}\alpha] = [u, (S^* + N^*)V_{jk}\alpha] \\ &\quad = [u, V_{jk}(\bar{\zeta}\alpha)] + \textstyle\sum_{s>k}\sum_r[u, V_{rs}(\bar{f}_{jkrs}\alpha)] = \displaystyle\int_{C^1} \zeta U_{jk}(u)(\zeta)\bar{\alpha}(\zeta)\,dm(\zeta) \\ &\qquad + \textstyle\sum_{s>k}\sum_r \displaystyle\int_{C^1} f_{jkrs}(\zeta)U_{rs}(u)(\zeta)\bar{\alpha}(\zeta)\,dm(\zeta). \end{aligned}$$

Since this last chain of equalities is valid for all α in $L^2(m)$ with bounded support, it follows that except on an m-null set,

$$U_{jk}(Tu)(\zeta) = \zeta U_{jk}(u)(\zeta) + \sum_{s>k}\sum_r f_{jkrs}(\zeta) U_{rs}(u)(\zeta).$$

To verify the properties of the decomposition, there remains only to show that the mappings F and G of Definition (1.1) are bounded mappings of L. Let U and V be the mappings of H into L and L into H defined by the double sequences $\{U_{jk}\}$ and $\{V_{jk}\}$. Then, it follows by a simple computation that $F = G = UNV$. Since U, V, and N are bounded, so therefore are F and G, and the proof is complete.

COROLLARY. *If T is a subnormal operator in the Hilbert space H, then T has a weak normal decomposition of the first order.*

Proof. By the argument of Lemma 2 of [7], T has a normal extension in a separable Hilbert space H_1 containing H as a closed subspace. The result then follows by the application of Theorems (1.2) and (1.3).

2. Let G be an open set of the Euclidean n-space E^n with coordinates $x = (x_1, \cdots, x_n)$. We denote by D_j the elementary differential operator $\partial/\partial x_j$ $(1 \leqq j \leqq n)$, while for the multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with α_j a non-negative integer for each j, we set $|\alpha| = \sum_j \alpha_j$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$.

Let L and B be two differential operators defined on G,

$$L = \sum a_\alpha(x) D^\alpha, \qquad B = \sum b_\beta(x) D^\beta, \tag{2.1}$$

with coefficients a_α and b_β which are infinitely differentiable functions on G. Let $C_c^\infty(G)$ be the family of infinitely differentiable functions with compact support in G, $(u, v) = \int_G u(x)\bar{v}(x)\,dx$, the inner product in $L^2(G)$.

If we topologize $C_c^\infty(G)$ in the fashion of L. Schwartz [19], we obtain a locally convex linear topological space. Its dual $\mathfrak{D}'(G)$ is the space of distributions on G. Each locally summable function u on G defines an element λ_u of $\mathfrak{D}'(G)$ by $\lambda_u(\psi) = (\psi, u)$ for ψ in $C_c^\infty(G)$.

An element λ of $\mathfrak{D}'(G)$ is said to be of order r at most, written $\lambda \in (C_c^r(G))'$, if there exists a constant $c > 0$ such that

$$|\lambda(\psi)| \leqq c \sup\{|D^\alpha\psi(x)|, x \in G, |\alpha| \leqq r\}$$

for all ψ in $C_c^\infty(G)$. It follows trivially that the same inequality persists for all ψ in $C_c^r(G)$. For each such λ, we define

$$\|\lambda\|_r' = \text{the least such constant } c.$$

We shall have occasion to apply Theorem XXII of [19], p. 86, which asserts that a set H of distributions on G is bounded in the topology of $\mathcal{D}'(G)$ if and only if for each open subset G_0 with compact closure in G, there exists an integer $r(G_0)$ and a constant $c(G_0) > 0$ such that

$$\| \lambda \|_{(C^{r(G_0)}(G_0))'} \leqq c(G_0), \tag{2.2}$$

for all λ in H. (In particular, each λ in H is of order $r(G_0)$ at most on G_0).

We shall assume throughout the rest of this paper that B is positive, i. e.

$$(Bu, u) > 0, \qquad u \in C_c^\infty(G). \tag{p_0}$$

The complex vector space $C_c^\infty(G)$ with the inner product $[u,v] = (Bu, v)$ is therefore a pre-Hilbert space, and we may complete it to a Hilbert space H_B with inner product $[u,v]$ and norm $\| u \|_B$.

LEMMA 2.1. *There exists a denumerable dense set in the linear topological space $C_c^\infty(G)$.*

Proof. We shall sketch the proof which is elementary in character. Let j be an element from $C_c^\infty(E^n)$ with $\int j(x)\, dx = 1$. Let F_0 be the family of finite linear combinations with rational coefficients of functions of the form $j_{\epsilon,y}(x) = \epsilon^{-n} j(\epsilon^{-1}(x - y))$, where ϵ is a positive rational number, y is a point of G all of whose coordinates are rational, and ϵ is so small that $j_{\epsilon,y}$ lies in $C_c^\infty(G)$. It is easy to see that for each u in $C_c^\infty(G)$, $u(x) = \int j_{\epsilon,y}(x) p(y)\, dy$ converges to zero in $C_c^\infty(G)$ as $\epsilon \to 0$, and that the integral can be approximated in $C_c^\infty(G)$ by its Riemann sums taken over points y_j with rational coordinates and with rational values assumed by the function $u(y)$.

COROLLARY. *H_B is a separable Hilbert space.*

In addition to (p_0), we make the following sharper assumption on the character of the space H_B.

(p_1) The identity mapping of $C_c(G)$ into $\mathcal{D}'(G)$ can be extended to a continuous injective mapping J of H_B into $\mathcal{D}'(G)$.

If (p_1) holds, as we shall assume henceforward, H_B can be continuously identified by J with a linear subset of $\mathcal{D}'(G)$. It follows from the argument of Section 2 of [17] that (p_1) is implied by the following stronger positivity assumption on the differential operator B:

(p_2) There exists a function p in $C^\infty(G)$ such that $(Bu, u) \geqq (pu, u)$ for all u in $C_c^\infty(G)$.

In the eigenfunction expansion theorems of Section 4, we shall assume only that (p_1) holds, but our conclusions may be strengthened somewhat if (p_2) also holds. In the latter case, J is a continuous mapping not only into $\mathfrak{D}'(G)$ but also into $L^2(p)$.

It follows from Theorem XXII of [19], as stated above, since the image of the unit ball in H_B under J is a bounded set in $\mathfrak{D}'(G)$, that for each open subset G_0 with compact closure in G, there exists an integer $r(G_0)$ and a positive constant $c(G_0)$ such that

$$\| Jv \|_{(C^{r(G_0)}(G_0))'} \leqq c(G_0) \| v \|_B \tag{2.3}$$

for all v in H_B.

In particular, if u is an element of $C_c^\infty(G)$ and G_0 is an open neighborhood of the support of u with compact closure in G, then

$$|(Lu, v)| \leqq c(G_0) \| Lu \|_{C^r(G_0)} \| v \|_B, \qquad r = r(G_0). \tag{2.4}$$

Thus for u fixed, (Lu, v) is a bounded conjugate-linear functional defined on the dense subset $C_c^\infty(G)$ of H_B. There exists, therefore, an unique element $A_0 u$ of H_B for which

$$[A_0 u, v] = (Lu, v), \qquad v \in C_c^\infty(G). \tag{2.5}$$

Definition 2.1. The operator A_0 with domain $C_c^\infty(G)$ in H_B is said to be the minimal realization of the pair of differential operators (L, B).

Let L' be the adjoint differential operator to L on G, defined by $L'u = \sum (-1)^{|\alpha|} D^\alpha (\bar{a}_\alpha(x) u)$ for u in $C_c^\infty(G)$. Let A'_0 be the minimal realization in H_B of the pair (L', B).

Definition 2.2. $A_1 = (A'_0)^*$ is said to be the maximal realization of the pair (L, B).

Definition 2.3. The operator A in H_B is said to be a realization of the pair of differential operators (L, B) if $A_0 \subseteq A \subseteq A_1$.

3. This section is devoted to establishing the following representation theorem for the general continuous linear mapping from an L^p space into $\mathfrak{D}'(G)$.

THEOREM 2.1. *Let m be a finite measure on the σ-algebra Ω of Borel sets of C^1, V a continuous linear mapping of $L^p(m)$ $(1 \leqq p < \infty)$ into $\mathfrak{D}'(G)$. Then there exists a bounded weakly measurable function f from C^1 to $\mathfrak{D}'(G)$ such that*

$$(V\alpha, u) = \int_{C^1} (f(\zeta), u) \alpha(\zeta) \, dm(\zeta), \tag{2.6}$$

for all u in $C_c^\infty(G)$ and all α in $L^p(m)$.

The distribution $f(\zeta)$ *may be chosen so that its order on each open subset* G_0 *with compact closure in* G *is bounded for all* ζ. *If* V *is a bounded linear mapping into* $L^2(p)$ *with* $p \in C^\infty(G)$, f *may be chosen of order* $(n+2)$ *at most on* G *for every* ζ *in* C^1.

Proof. Without loss of generality, we may restrict ourselves to an open subset G_0 with compact closure in G. Indeed, if $f_1(\zeta)$ and $f_2(\zeta)$ are functions satisfying the conclusion of the theorem for the open sets G_1 and G_2 respectively, (2.6) implies that

$$\int_{C^1} (f_1(\zeta), u)\,\alpha(\zeta)\,dm(\zeta) = \int_{C^1} (f_2(\zeta), u)\,\alpha(\zeta)\,dm(\zeta) \tag{2.7}$$

for all u in $C_c^\infty(G_1 \cap G_2)$ and all α in $L^p(m)$. It follows that $(f_1(\zeta), u) = (f_2(\zeta), u)$ for ζ outside a m-null set depending on u. If F_0 is the dense denumerable set in $C_c^\infty(G)$ constructed in Lemma (2.1), it suffices for the equality to hold for u in F_0 in order for it to hold for all u in $C_c^\infty(G)$. Thus $(f_1(\zeta), u) = (f_2(\zeta), u)$ for all u in $C_c^\infty(G)$ for ζ in the complement of a fixed null set. Since $f(\zeta)$ may be taken zero on any m-null set without affecting the validity of (2.6), it follows that the distributions $f_1(\zeta)$ and $f_2(\zeta)$ may be taken equal on $G_1 \cap G_2$ for all ζ. In particular, if $G_1 \subset G_2$, $f_1(\zeta)$ and $f_2(\zeta)$ coincide on G_1 for all ζ. Choosing an increasing sequence G_n of open subsets with compact closure in G whose union equals G, we obtain the desired function for G by taking the common value of $f_m(\zeta)$ on G_n for $m > n$.

Let G_0' be an open subset with compact closure in G such that $\bar{G}_0 \subset G_0'$. On G_0', applying Theorem XXII of [19] once more, the image of $L^p(m)$ under V is contained in the subfamily of distributions of order $\leqq r$, and for all α in $L^p(m)$, we have

$$\| V\alpha \|_{(C^r(G_0'))'} \leqq c \, \| \alpha \|_{L^p(m)}. \tag{2.7}$$

Let R^n be the dual space to E^n, with the pairing $\langle x, \xi \rangle = \sum_j x_j \xi_j$ for x in E^n, ξ in R^n. For $s > n/2$, we define

$$e_s(x) = \int_{R^n} (|\xi|^2 + 1)^{-s} e^{-i\langle x, \xi \rangle} d\xi. \tag{2.8}$$

The function $e_s(x)$, which is infinitely differentiable for $x \neq 0$, is a fundamental solution for the elliptic differential operator $Q_s = (-\Delta + 1)^s$, where Δ is the Laplace operator $(\Delta = \sum_j D_j^2)$. Let $\epsilon > 0$ be smaller than the distance from G_0 to the complement of G_0', and let $q(x)$ be a function from $C_c^\infty(E^n)$ which is equal to 1 for $|x| < \epsilon/2$, and equal to zero for $|x| \geqq \epsilon$.

Let $e_1(x)=q(x)e(x)$, $e_2(x)=e(x)-e_1(x)$, $e_3(x)=Q_se_2(x)$. Since $e_2(x)=0$ in a neighborhood of $x=0$, e_2 and e_3 lie in $C^\infty(E^n)$.

Suppose that λ is a distribution from $(C^r(G_0'))'$. For z in G_0, $e_1(x-z)$ considered as a function of x has its support in G_0'. Suppose that $2s>r+n$. Then differentiating (2.8) under the integral sign, we see that $e(x)$ lies in $C^r(E^n)$, and hence $e_1(x-z)$ considered as a function of x, $e_{1,z}(x)$, lies in $C_c^r(G_0')$. Further, $e_{1,z}$ considered as an element of $C_c^r(G_0')$ varies continuously with z for z in G_0.

It follows immediately that $h(z)=(\lambda,e_{1,z})$ is a well-defined, uniformly bounded, continuous function of z in G_0. For u in $C_c^\infty(G_0)$, we have, moreover,

$$(2.9)\qquad (h,Q_su)=\int_{G_0}(\lambda,e_{1,z})Q_su(z)\,dz=(\lambda_x,\int_{G_0}e_1(x-z)Q_su(z)\,dz)$$
$$=(\lambda,u)-(\lambda_x,\int_{G_0}e_3(x-z)u(z)\,dz).$$

If we set $k(z)=(\lambda_x,e_3(x-z))$, the last term in (2.9) is equal to (k,u). Since $e_3(x-z)$ has its support in G_0' for z in G_0 and is infinitely differentiable, $k(z)$ is an infinitely differentiable function which is uniformly bounded on G_0 with a bound depending on the $(C^r(G_0'))'$-norm of λ.

Let H and K be the bounded linear mappings of $(C^r(G_0'))'$ into $C^0(G_0)$ defined by $H\lambda=\kappa$, $K\lambda=k$, respectively. Then (2.9) can be written using distribution derivatives in the form

$$(2.10)\qquad \lambda=Q_sH\lambda+K\lambda.$$

Let W and Z be the mappings of $L^p(m)$ into $C^0(G_0)$ defined by $W\alpha=HV\alpha$, $Z\alpha=KV\alpha$. By the Dunford-Pettis theorem [8], there exist functions $w(x,\zeta)$, $z(x,\zeta)$ measurable with respect to the product of Lebesgue measure on G_0 and m on C^1 such that

$$(2.11)\qquad (W\alpha)(x)=\int_{C^1}w(x,\zeta)\alpha(\zeta)\,dm(\zeta);\qquad x\in G_0,\ \alpha\in L^p(m),$$

$$(2.12)\qquad (Z\alpha)(x)=\int_{C^1}z(x,\zeta)\alpha(\zeta)\,dm(\zeta);\qquad x\in G_0,\ \alpha\in L^p(m),$$

while there exists a constant c_0 such that for all x in G_0,

$$(2.13)\qquad \|w(x,-)\|_{L^{p'}(m)}\leqq c_0,\qquad \|z(x,-)\|_{L^{p'}(m)}\leqq c_0.$$

(p' = the conjugate exponent to p).

From (2.12) and the Fubini theorem (using the boundedness of G_0), we see that for ζ in the complement of a set M_0 of m-measure zero, $w(x,\zeta)$

and $z(x,\zeta)$, considered as functions of x with ζ held fixed, yield elements of $L^{p'}(G_0)$ which we designate as $w(\zeta)$ and $z(\zeta)$. For ζ in M_0, we set $w(\zeta)=0$, $z(\zeta)=0$. In terms of these functions from C^1 to $L^{p'}(G_0)$, (2.12) and (2.13) become

$$(2.14)\qquad W\alpha=\int_{C^1} w(\zeta)\alpha(\zeta)\,dm(\zeta); \qquad Z\alpha=\int_{C^1} z(\zeta)\alpha(\zeta)\,dm(\zeta).$$

Finally, we see from (2.10) that $V\alpha=Q_sW\alpha+Z\alpha$. By a simple argument on the convergence of integrals of distributions,

$$(2.15)\qquad V\alpha=\int_{C^1}(Q_sw(\zeta)+z(\zeta))\alpha(\zeta)\,dm(\zeta).$$

Equation (2.15) is equivalent to (2.6) if we set $f(\zeta)=Q_sw(\zeta)+z(\zeta)$. The distribution $f(\zeta)$ is of order $2s$ at most on G_0. If V is a continuous linear mapping into $L^2(p)$, then s may be chosen equal to $[n/2]+1$ and $f(\zeta)$ is of order $n+2$ at most on G_0.

4. Let A be a realization of the pair (L,B) in the Hilbert space H_B. We note from the definitions of the minimal and maximal realizations that $C_c^\infty(G)\subset D(A)\cap D(A^*)$.

THEOREM 4.1. *For each decomposition of a decomposable realization A of (L,B) in the Hilbert space H_B, there exists an eigenfunction expansion in the sense of Definition 1 of the Introduction. If the decomposition is of finite order, the corresponding eigenfunction expansion is regular. If the decomposition is normal, the corresponding eigenfunction expansion is normal and $e_{jk}(\zeta)=f_{jk}(\zeta)$.*

Proof. Let $(m,\{f_{jkrs}\},\{g_{jkrs}\},\{U_{jk}\},\{V_{jk}\})$ be a permissible family defining a decomposition of A. We recall that by Definition (1.1),

$$L=\textstyle\sum_{jk}\oplus(L^2(m))_{jk},$$

F and G are bounded linear transformations of L defined by

$$(4.1)\qquad F(\{\alpha_{rs}\})_{jk}=\textstyle\sum_{s>k}\sum_r f_{jkrs}\alpha_{rs},$$

$$(4.2)\qquad G(\{\alpha_{rs}\})_{jk}=\textstyle\sum_{s>k}\sum_r g_{jkrs}\alpha_{rs}.$$

U is the bounded linear mapping of H_B into L defined by $(Uu)_{jk}=U_{jk}u$, while V is the bounded linear mapping of L into H_B with

$$(4.3)\qquad V(\{\alpha_{rs}\})=\textstyle\sum_{r,s}(\alpha_{rs}).$$

Let B_{rs} be the injection mapping of $L^2(m)$ into the (r,s)-th component of L. For each positive integer r, we define $T_r = F^rU$, a bounded linear mapping of H_B into L. If $T_{r,jk} = B_{jk}{}^*T_r$ is the projection of T_r on its (j,k)-th component, $T_{r,jk}$ is a bounded linear mapping of H_B into $L^2(m)$. Let $T'_{r,jk} = VG^rB_{jk}$, a bounded linear mapping of $L^2(m)$ into H_B.

In terms of the above definitions, properties (b) and (c) of Definition (1.2) become, respectively:

$$U_{jk}(Au)(\zeta) = \zeta U_{jk}(u)(\zeta) + T_{1,jk}(u)(\zeta), \tag{4.4}$$

$$[V_{jk}(\alpha), A^*u] = [V_{jk}(\zeta\alpha), u] + [T'_{1,jk}(\alpha), u], \tag{4.5}$$

for u in $C_c{}^\infty(G)$, α in $L^2(m)$ with bounded support.

For the same class of α and u, we obtain by applying properties (b) and (c) of Definition (1.2) to $T_{r,jk}$ and $T'_{r,jk}$ themselves,

$$T_{r,jk}(Au)(\zeta) = \zeta T_{r,jk}(u)(\zeta) + T_{r+1,jk}(u)(\zeta), \tag{4.6}$$

$$[T'_{r,jk}(\alpha), A^*u] = [T'_{r,jk}(\zeta\alpha), u] + [T'_{r+1,jk}(\alpha), u]. \tag{4.7}$$

We remark that equations (4.4) and (4.6) hold for ζ in the complement of an m-null set $M(u)$. By Lemma (2.1), however, $M(u)$ may be chosen independent of u, since we may take $M = \bigcup_t M(u_t)$ for a dense denumerable family $\{u_t\}$ in $C_c{}^\infty(G)$, and for ζ in the complement of M, (4.4) and (4.6) will hold for all u in $C_c{}^\infty(G)$ by continuity.

For $r \geqq k$, $T'_{r,jk} = 0$. If the decomposition is of finite order r_0 $T_r = 0$ for $r \geqq r_0$.

We now proceed to the application of Theorem (3.1). The continuous linear mappings $U_{jk}{}^*$, V_{jk}, $T_{r,jk}{}^*$, and $T'_{r,jk}$ of $L^2(m)$ into H_B may be considered, by property (p_1) of Section 2, as continuous linear mappings of $L^2(m)$ into $\mathfrak{D}'(G)$. Hence there exist functions $f_{jk}(\zeta)$, $e_{jk}(\zeta($, $t'_{r,jk}(\zeta)$ from $C_2{}^1$ to $\mathfrak{D}'(G)$, as described in Theorem (3.1), such that for α in $L^2(m)$,

$$U_{jk}{}^*(\alpha) = \int_{C^1} f_{jk}(\zeta)\alpha(\zeta)\,dm(\zeta), \tag{4.8}$$

$$V_{jk}(\alpha) = \int_{C^1} e_{jk}(\zeta)\bar{\alpha}(\zeta)\,dm(\zeta), \tag{4.9}$$

$$T_{r,jk}{}^*(\alpha) = \int_{C^1} t_{r,jk}(\zeta)\alpha(\zeta)\,dm(\zeta), \tag{4.10}$$

$$T'_{r,jk}(\alpha) = \int_{C^1} t'_{r,jk}(\zeta)\alpha(\zeta)\,dm(\zeta). \tag{4.11}$$

For u in $C_c^\infty(G)$, α in $L^2(m)$,

$$\int_{C^1} U_{jk}(u)(\zeta)\bar{\alpha}(\zeta)dm(\zeta) = [u, U_{jk}^*(\alpha)] = (Bu, \int_{C^1} f_{jk}(\zeta)\bar{\alpha}(\zeta)dm(\zeta)$$
$$= \int_{C^1} (Bu, f_{jk}(\zeta))\bar{\alpha}(\zeta)dm(\zeta),$$

the equality of the first with the last term in the chain of equations implying that

$$U_{jk}(u)(\zeta) = (Bu, f_{jk}(\zeta)) \tag{4.12}$$

for ζ in the complement of an m-null set M. Similarly, the equations

$$\int_{C^1} U_{jk}(Au)(\zeta)\bar{\alpha}(\zeta)dm(\zeta) = [Au, U_{jk}^*(\alpha)] = (Lu, \int_{C^1} f_{jk}(\zeta)\alpha(\zeta)dm(\zeta))$$
$$= \int_{C^1} (Lu, f_{jk}(\zeta))\bar{\alpha}(\zeta)dm(\zeta)$$

for u in $C_c^\infty(G)$, α in $L^2(m)$, imply that

$$U_{jk}(Au)(\zeta) = (Lu, f_{jk}(\zeta)) \tag{4.13}$$

in the complement of an m-null set. Parallel arguments for $T_{r,jk}$ *yield*

$$T_{r,jk}(u)(\zeta) = (Bu, t_{r,jk}(\zeta)), \tag{4.14}$$

$$T_{r,jk}(Au)(\zeta) = (Lu, t_{r,jk}(\zeta)) \tag{4.15}$$

for all u in $C_c^\infty(G)$ and ζ in the complement of an m-null set.

The equations (4.4) and (4.6) become, respectively,

$$(Lu, f_{jk}(\zeta)) = \zeta(Bu, f_{jk}(\zeta)) + (Bu, t_{1,jk}(\zeta)), \tag{4.16}$$

$$(Lu, t_{r,jk}(\zeta)) = \zeta(Bu, t_{r,jk}(\zeta)) + (Bu, t_{r+1,jk}(\zeta)), \tag{4.17}$$

for all u in $C_c^\infty(G)$ and ζ in the complement of an m-null set M. Since we may alter any of the functions f_{jk}, e_{jk}, $t_{r,jk}$, and $t'_{r,jk}$ on an m-null set without affecting the validity of any of the previous equations, we may set all of them equal to zero on M. Then (4.16) and (4.17) are valid for all ζ in C^1. In terms of distribution derivatives, they may be written as

$$L'f_{jk}(\zeta) - \bar{\zeta}Bf_{jk}(\zeta) = Bt_{1,jk}(\zeta) \tag{4.16$'$}$$

$$L't_{r,jk}(\zeta) - \bar{\zeta}Bt_{r,jk}(\zeta) = Bt_{r+1,jk}(\zeta). \tag{4.17$'$}$$

If the decomposition is of finite order r_0, we have $t_{r,jk} = 0$ for $r \geqq r_0$, and f_{jk} is an eigenfunction of order r_0 of (L', B) with eigenvalue $\bar{\zeta}$.

Similarly, equations (4.5) and (4.7) written in terms of the functions $e_{jk}(\zeta)$ and $t'_{r,jk}(\zeta)$ become for u in $C_c^\infty(G)$,

$$(4.18) \qquad (e_{jk}(\zeta), L'u) = \zeta(e_{jk}(\zeta), Bu) + (t'_{1,jk}(\zeta), Bu),$$

$$(4.19) \qquad (t'_{r,jk}(\zeta), L'u) = \zeta(t'_{r,jk}(\zeta), Bu) + (t'_{r+1,jk}(\zeta), Bu),$$

which, after a change of the functions e_{jk} and $t'_{r,jk}$ on a m-null set independent of u, are valid for all ζ in C^1. In terms of distribution derivatives, (4.18) and (4.19) may be written as

$$(4.18)' \qquad Le_{jk}(\zeta) - \zeta Be_{jk}(\zeta) = Bt'_{1,jk}(\zeta),$$

$$(4.19)' \qquad Lt'_{r,jk}(\zeta) - \zeta Bt'_{r,jk}(\zeta) = Bt'_{r+1,jk}(\zeta).$$

Since $t'_{r,jk}(\zeta)$ may be chosen null for all ζ in C^1 if $r \geqq k$, it follows that $e_{jk}(\zeta)$ is an eigenfunction of order k of (L, B) with eigenvalue ζ.

Finally, if $V_{jk} = U_{jk}^*$, $e_{jk}(\zeta) = f_{jk}(\zeta)$ for all ζ, and the decomposition is normal.

Thus all of the conclusions of the theorem have been verified, since the functions $\{e_{jk}(\zeta)\}$ and $\{f_{jk}(\zeta)\}$ yield an eigenfunction expansion in the sense of Definition 1 which is regular if the decomposition is of finite order and normal if the decomposition is normal.

THEOREM 4.2. *To each weak normal decomposition of finite order of a realization A of (L, B), there corresponds a finite measure m on Ω and a double sequence $\{f_{jk}(\zeta)\}$ of functions from C^1 to $\mathfrak{D}'(G)$ such that for each u in $C_c^\infty(G)$, $c_{jk}(\zeta) = (Bu, f_{jk}(\zeta))$ lies in $L^2(m)$, the mapping U defined by $U_{jk}u = c_{jk}$ defines an isometry of H_B into L, for each u in H_B,*

$$u = \sum_{j,k} \int_{C^1} f_{jk}(\zeta) U_{jk}(u)(\zeta)\, dm(\zeta),$$

while $f_{jk}(\zeta)$ is an eigenfunction of (L', B).

Proof. The present conclusions follow immediately from the portion of the preceding proof that remains valid for a weak normal decomposition.

5. To derive eigenfunction expansions in the classical sense from the theorems of the previous section, we must restrict ourselves to a subclass of pairs (L, B) for which the distribution eigenfunctions are ordinary differentiable functions. The simplest, and the widest such subclass, is of course that covered by the following definition.

Definition 5.1. The pair of differential operators (L, B) is said to be hypoelliptic in the widest sense if every distribution solution u of the equation $(L - \zeta B)u = v$ with v in $C^\infty(G)$, is itself an infinitely differentiable function in G.

It is an immediate consequence of Definition (5.1) that for a pair (L, B) which is hypoelliptic in the widest sense, every eigenfunction of arbitrary order of (L, B), for arbitrary eigenvalue ζ, is an infinitely differentiable function.

To obtain further regularity properties for the eigenfunction expansions obtained in the theorems of Section 4, we introduce the spaces $W^{r,2}(\dot{G}_0)$ and the (r, G_0)-norm, where G_0 is an open subset of G and r is an integer. For $r \geqq 0$,

$$W^{r,2}(G_0) = \{u : D^\alpha u \in L^2(G_0) \text{ for } |\alpha| \leqq r\}.$$

The corresponding norm is given by $(\| u \|_{r,G_0})^2 = \int_{G_0} \sum_{|\alpha| \leqq r} |D^\alpha u|^2 \, dx$. For $r < 0$, $W^{r,2}(G_0) = \{u : |(u, \psi)| \leqq c(u) \| \psi \|_{r,G_0}, \psi \in C_c^\infty(G_0)\}$. The corresponding norm is given by

$$\| u \|_{r,G_0} = \sup\{|(u, \psi)| : \psi \in C_c^\infty(G_0), \| \psi \|_{-r,G_0} \leqq 1\}.$$

Definition 5.2. The pair of differential operators (L, B) is said to be hypoelliptic in the strong sense if given two open subsets G_0 and G_0' with compact closure in G, $\bar{G}_0 \subseteq G_0'$, and an integer j, there exists a constant $c(G_0, G_0', j) > 0$ such that if u is a distribution solution of the equation $Lu = Bv$ with $\| u \|_{j,G_0'} < \infty$, $\| v \|_{j,G_0'} < \infty$, then

$$\| u \|_{j+1,G_0} \leqq c(G_0, G_0', j)\{\| u \|_{j,G_0'} + \| v \|_{j,G_0'}\} < \infty. \tag{5.1}$$

THEOREM 5.1. *Let (L, B) and (L', B) be pairs of differential operators hypoelliptic in the strong sense on G, A a decomposable realization of (L, B). Then the functions $\{e_{jk}(\zeta)\}$ and $\{f_{jk}(\zeta)\}$ obtained in Theorem (4.1) may be chosen infinitely differentiable for all ζ in C^1. Further, for every open subset G_0 with compact closure in G, bounded Borel set σ in C^1, and positive integers r and k, there exists a positive constant $c(G_0, \sigma, r, k)$ such that*

$$\begin{aligned} &\int_\sigma |D^\alpha e_{jk}(x, \zeta)|^2 \, dm(\zeta) \leqq c(G_0, \sigma, r, k), \\ &\int_\sigma |D^\alpha f_{jk}(x, \zeta)|^2 \, dm(\zeta) \leqq c(G_0, \sigma, r, k) \end{aligned} \tag{5.2}$$

for all x in G_0, $|\alpha| \leqq r$.

Proof. It follows from the proof of Theorem (3.1) that there exists a positive integer s, depending only on G_0 and on the imbedding of H_B in $\mathfrak{D}'(G)$, and sequences of functions $\{p_{jk}(x,\zeta)\}$, $\{q_{jk}(x,\zeta)\}$, $\{p_{r,jk}(x,\zeta)\}$, $\{q_{r,jk}(x,\zeta)\}$ such that

$$\begin{aligned} e_{jk}(x,\zeta) &= (-\Delta+1)^s p_{jk}(x,\zeta), \\ f_{jk}(x,\zeta) &= (-\Delta+1)^s q_{jk}(x,\zeta), \\ t'_{r,jk}(x,\zeta) &= (-\Delta+1)^s p_{r,jk}(x,\zeta), \\ t_{r,jk}(x,\zeta) &= (-\Delta+1)^s q_{r,jk}(x,\zeta), \end{aligned} \tag{5.3}$$

while there exists a constant $c(G_0,B)$ such that

$$\begin{aligned} &\int_{C^1} | p_{jk}(x,\zeta)|^2\, dm(\zeta) \leqq c(G_0,B) \| V_{jk} \|^2, \\ &\int_{C^1} | q_{jk}(x,\zeta)|^2\, dm(\zeta) \leqq c(G_0,B) \| U_{jk} \|^2, \\ &\int_{C^1} | p_{r,jk}(x,\zeta)|^2\, dm(\zeta) \leqq c(G_0,B) \| T'_{r,jk} \|^2, \\ &\int_{C^1} | q_{r,jk}(x,\zeta)|^2\, dm(\zeta) \leqq c(G_0,B) \| T_{r,jk} \|^2, \end{aligned} \tag{5.4}$$

for all x in G_0. Noting by Definition (1.1) that $\| U_{jk} \|$ and $\| V_{jk} \|$ are uniformly bounded for all j and k, and that $\| T'_{r,jk} \| \leqq c(r)$, $\| T_{r,jk} \| \leqq c(r)$, we obtain from (5.3), (5.4), and the definition of the negative norms given above.

$$\begin{aligned} &\int_{C^1} \| e_{jk}(\zeta) \|^2_{-2s,G_0}\, dm(\zeta) \leqq c_0, \\ &\int_{C^1} \| f_{jk}(\zeta) \|^2_{-2s,G_0}\, dm(\zeta) \leqq c_0, \\ &\int_{C^1} \| t'_{r,jk}(\zeta) \|^2_{-2s,G_0}\, dm(\zeta) \leqq c(r), \\ &\int_{C^1} \| t_{r,jk}(\zeta) \|^2_{-2s,G_0}\, dm(\zeta) \leqq c(r), \end{aligned} \tag{5.5}$$

with the integrands in each of the integrals of (5.5) finite for ζ in the complement of an m-null set. If we set all the various functions equal to zero for ζ in the m-null set concerned, the integrands will be finite for all ζ in C^1. From the proof of Theorem (4.1), we have

$$
(5.6)\qquad
\begin{aligned}
Le_{jk}(\zeta) &= \zeta B e_{jk}(\zeta) + Bt'_{1,jk}(\zeta),\\
Lt'_{r,jk}(\zeta) &= \zeta Bt'_{r,jk}(\zeta) + Bt'_{r+1,jk}(\zeta),\\
L'f_{jk}(\zeta) &= \bar{\zeta} Bf_{jk}(\zeta) + Bt_{1,jk}(\zeta),\\
L't_{r,jk}(\zeta) &= \bar{\zeta} Bt_{r,jk}(\zeta) + Bt_{r+1,jk}(\zeta).
\end{aligned}
$$

It follows from the Definition (5.2) of hypoellipticity in the strong sense, by a simple inductive argument starting from the finiteness of the $(-2s, G_0)$-norms of all the functions concerned, that the (r, G_0'')-norm of each function is finite for all r and every G_0'' with compact closure in G_0. It follows from a well-known theorem of Sobolev ([19], vol. 2) that the functions $e_{jk}(x,\zeta)$ and $f_{jk}(x,\zeta)$ are infinitely differentiable in x in G_0 for every ζ in C^1.

The inequalities (5.2) follow from (5.5), (5.6), and Definition (5.2) by a direct inductive argument.

Remark. Conditions for the hypoellipticity of (L,B) in terms of the characteristic forms of the operators L and B may be obtained from the results of [14] and [6]. If L is elliptic and B is of order less than the order of L, then (L,B) is hypoelliptic in the strong sense. Since every non-degenerate ordinary differential operator is elliptic, the latter class includes all pairs (L,B) treated in the theory for ordinary differential operators.

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SINGULAR INTEGRALS IN TWO DIMENSIONS.*

By JOHN J. MCKIBBEN.[1]

1. Introduction. We recall that a real-valued C^∞ function $u = (x_1, x_2)$ of the two real variables x_1 and x_2 is said to be "rapidly decreasing" if the absolute value of the function $P(x_1, x_2)Du$ is bounded on R^2 for all real polynomials $P(x_1, x_2)$ and all partial derivatives Du of u. The space of all C^∞ rapidly decreasing function on R^2 is topologized according to the method of Laurent Schwartz [6], and the resulting space will be denoted by (ζ). A "tempered distribution" is a continuous linear mapping of (ζ) into the real numbers.

It is well-known that every function which is locally square integrable is a distribution [6]. On the other hand, as this paper is written, it appears to be unknown whether or not even some of the simplest functions which are not locally square integrable—for example, the reciprocals of polynomials of several variables—are distributions. We prove here that the reciprocal of any polynomial in two real variables is a distribution and that it is, in fact, a tempered distribution. More significant, perhaps, is the fact that the proof we give is entirely constructive and yields a method for computing the reciprocal of any polynomial in two variables. As the author will show in another paper, a variant of the same method allows one to construct the reciprocal of an arbitrary analytic function of two real variables.

It may be useful to express the analytical content of our main result in terms of the more widely known, if somewhat more obscure, idea of a "partie finie" of a divergent integral. Rephrased, the theorem says that if $q(\eta_1, \eta_2)$ is any polynomial in two variables, it is possible to define in the sense of Hadamard the expression

$$Pf. \int_{R^2} u(\eta_1, \eta_2) [q(\eta_1, \eta_2)]^{-1} \, d\eta_1 d\eta_2.$$

In our proof, we follow Hadamard's procedure of integrating by parts and then discarding the boundary integrals which tend to become infinite.

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We also mention an application to the theory of partial differential equations. Define the polynomial $Q(\eta_1, \eta_2)$ by setting

$$Q(\eta_1, \eta_2) = q(-2\pi i\eta_1, -2\pi i\eta_2)$$

and let E be a tempered distribution reciprocal to Q, so that $EQ = 1$ (the existence of E is insured by our main result). Let F denote the Fourier transform operator

$$[Fu](x_1, x_2) = \int_{R^2} \exp[-2\pi i(x_1\eta_1 + x_2\eta_2)] u(\eta_1, \eta_2)\, d\eta_1 d\eta_2,$$

for any $u \in (\zeta)$, and write $q(D)$ for the operation $q(\partial/\partial\eta_1, \partial/\partial\eta_2)$, $q(-D)$ for the operator $q(-\partial/\partial\eta_1, -\partial/\partial\eta_2)$. Then

$$[Fq(-D)u](x_1, x_2) = Q(x_1, x_2) \cdot \{[Fu](x_1, x_2)\}. \tag{1.1}$$

Define the tempered distribution S by the formula $S(u) = E(Fu)$, $u \in (\zeta)$. Then, by the way differentiation of a distribution is defined and by (1.1),

$$[q(D)S](u) = S[q(-D)u] = E(Fq(-D)u) = E(QFu) = [EQ](Fu).$$

But $EQ = 1$ and since the distribution "1" at any function $w(x_1, x_2)$ of (ζ) takes the value $\int_{R^2} w\, dx_1 dx_2$, we obtain, finally, by the Fourier inversion formula, that $[EQ](Fu) = \int_{R^2} F(u)\, dx_1 dx_2 = u(0)$. Therefore

$$q(D)S = \delta, \tag{1.2}$$

where δ is the Dirac distribution. Equation (1.2) says that S is an elementary solution for the operator $q(D)$. It follows, therefore, that in two dimensions every partial differential operator with constant coefficients has a tempered elementary solution.

This result improves (in the case of two dimensions) an earlier one due to Malgrange [3] and Ehrenpreis [1], who showed that any partial differential operator with constant coefficients in any number of variables possesses an elementary solution. The question was left open whether or not, as L. Schwartz originally conjectured, there exists a tempered elementary solution. Also, the elegant existence proofs of Malgrange and Ehrenpreis use the Hahn-Banach Theorem and give no indication how, for a given operator, its elementary solution may be constructed. Hörmander [2] has given an explicit formula for these untempered elementary solutions, but one which again is non-constructive (involving the analytic continuation of a Fourier transform). Therefore it appears that our method is the first, even in the

low dimension 2, which allows one to construct completely any kind of an elementary solution.

A different proof of these results was part of the author's doctoral dissertation at the University of Chicago [4]. The author would like to express his gratitude to Professor M. S. Stone for his long-continued help and interest.

Since this paper was written there has appeared a proof that the reciprocal of an analytic function of n real variables is a distribution (see M. S. Lojasiewicz, "Division d'une distribution par une function analytique de variables reelles," Comptes Rendus de l'academie des sciences de Paris, Tome 246, No. 5, February 1958). The general proof is unconstructive. It seems unlikely that it will ever be possible to construct the reciprocal of the most general analytic function (or even polynomial) in a number of dimensions greater than two.

2. Notation. Let $\eta = (\eta_1, \eta_2)$ be a real vector and let $q = q(\eta_1, \eta_2)$ be a polynomial in the variables η_1, η_2. We assume throughout this paper that the coefficients of q are real numbers and that q has degree greater than zero.[2] Let V be the real variety given by the equation $q(\eta_1, \eta_2) = 0$.

A point η^0 of V will be called unexceptional in case there exists a neighborhood U of η^0 and a pair of functions $v_1(\eta), v_2(\eta)$, analytic and with strictly positive Jacobian (with respect to η_1, η_2) in U, such that q is identically equal to some integral power of v_1 in U. A point of V not satisfying these requirements is called exceptional.

We say that q is irreducible if no polynomial of positive degree and with complex coefficients divides q; otherwise, we say q is reducible. A point η of V is called singular if both first partial derivatives of q vanish at η. Then, clearly, if q is irreducible, a point of V is exceptional if and only if it is a singular point.

If q is reducible, let $q = \prod_{i=1}^{\nu} [q_i]^{\alpha_i}$ be its decomposition into integral powers α_i of distinct irreducible polynomials q_i. Then it is easy to see that a point of V is exceptional if and only if it is either a singular point of some variety $q_i = 0$, $(i = 1, \cdot \cdot, \nu)$, or else belongs to the intersection of two or more of these varieties.

[2] The problem of finding the reciprocal of a complex polynomial Q can be reduced to the corresponding problem in the real case by means of the identity $Q^{-1} = Q^* \mid Q \mid^{-2}$, where Q^* is the conjugate, $\mid Q \mid$ the absolute value, of Q.

Assume now that we have chosen coordinates η_1, η_2 in such a way that their origin does not lie on V, and define polynomials $p, p_1, \cdots, p_\nu$ by setting $p(\eta_1, \eta_2, \lambda) \equiv q(\lambda\eta_1, \lambda\eta_2)$, $p_i(\eta_1, \eta_2, \lambda) \equiv q_i(\lambda\eta_1, \lambda\eta_2)$, $(i = 1, \cdots, \nu)$, and let

$$H(\eta_1, \eta_2) \equiv \prod_{\substack{i,j=1 \\ i<j}}^{\nu} R_{ij} D_j, \tag{2.1}$$

where R_{ij} is the resultant of p_i and p_j, D_j the discriminant of p_j, where p_i and p_j are considered as polynomials in the variable λ. Since R_{ij} and D_j are homogeneous polynomials in the variables η_1 and η_2, the variety $H = 0$ is a cone with vertex at the origin. It is easy to see that every exceptional point of V will lie on the cone $H = 0$ as well as all points η^0 of V such that the line joining η^0 to the origin is tangent to V at η^0.

In order to save words later on, it is convenient to make the following convention. Let $g(\epsilon)$ be any complex-valued function defined in the interval $0 < \epsilon < \tau$. We shall write $g(\epsilon) = \infty(\epsilon)$ in case there exists a finite set M of negative fractions, complex numbers a and a_μ, $(\mu \in M)$, and a complex function $f(\epsilon)$ defined in the same interval as g such that $\lim_{\epsilon \to +0} f(\epsilon)$ exists and is finite and such that the equation

$$g(\epsilon) \equiv \sum_{\mu \in M} a_\mu \epsilon^\mu + a \log \epsilon + f(\epsilon) \tag{2.2}$$

hold for $0 < \epsilon < \tau$.

In case $g(\epsilon) = \infty(\epsilon)$, the complex number $\lim_{\epsilon \to +0} f(\epsilon)$ is uniquely determined and we shall denote it by $Pf.\, g(0)$.

3. A preliminary theorem.

THEOREM (3.1). *Let $q(\eta_1, \eta_2)$ be a real polynomial in the two real variables η_1, η_2 and let V be the variety $q = 0$. Let G be a region of R^2 the closure of which contains no exceptional points of V and let $G(\epsilon)$, $\epsilon > 0$, be that part of G where $|q| > \epsilon$. Then, for any u in (ζ),*

$$\int_{G(\epsilon)} u(\eta)[q(\eta)]^{-1} d\eta_1 d\eta_2 = \infty(\epsilon). \tag{3.2}$$

Proof. By means of a partition of unity, the integral on the left side of equation (3.2) is expressible as a sum of integrals of the form

$$\int_{U(\epsilon)} w(\eta)[q(\eta)]^{-1} d\eta_1 d\eta_2, \tag{3.3}$$

where $w(\eta)$ is a C^∞ function and where $U(\epsilon)$ is the intersection of $G(\epsilon)$

and a neighborhood U which is either disjoint from V or satisfies the conditions described in the definition of an unexceptional point. In the first case, the integral (3.3) has a finite limit. In the second, there exist coordinates v_1, v_2 in U such that $q \equiv (v_1)^a$ for some positive integer a. Setting

$$h(\epsilon) = \int_{W(\epsilon)} w(\eta(v))J(v)\,dv_1,$$

where $W(\epsilon)$ is the part of the level surface $|q| = \epsilon$ lying in U and where $J(v)$ is the Jacobian of the coordinates $\eta = (\eta_1, \eta_2)$ with respect to the coordinates $v = (v_1, v_2)$, we get that

$$\int_{U(\epsilon)} w(\eta)[q(\eta)]^{-1}\,d\eta_1 d\eta_2 = \int_{\epsilon}^{\infty} h(\epsilon)\epsilon^{-a}\,d\epsilon.$$

Since $h(\epsilon)$ is C^∞, we may integrate by parts on the right side of the last equation and thereby obtain an expansion of the form (2.2). It follows that the integral on the left side of equation (3.2) is $\infty(\epsilon)$ since it is a sum of expressions each of which is $\infty(\epsilon)$. This proves Theorem (3.1).

4. The two main theorems. Let us again assume that $q(0,0) \neq 0$ and define the homogeneous polynomial $H(\eta_1, \eta_2)$ by equation (2.1). If we introduce polar coordinates ρ and θ in the usual way in the $\eta_1\eta_2$-plane, the cone $H = 0$ will be given by equations $\theta = \theta_1, \theta = \theta_2, \cdots, \theta = \theta_\Gamma$. If δ is a positive number, we let the region $G(\delta)$ consist of all points $(\rho\cos\theta, \rho\sin\theta)$ such that $|\theta - \theta_i| > \delta$, $(i = 1, 2, \cdots, \Gamma)$, and if ϵ is a second positive number we denote by $G(\delta, \epsilon)$ that part of $G(\delta)$ where $|q| > \epsilon$. For $u \in (\zeta)$, set

$$g_\delta(u;\epsilon) = \int_{G(\delta,\epsilon)} u(\eta)[q(\eta)]^{-1}d\eta. \tag{4.1}$$

By Theorem (3.1), $g_\delta(u;\epsilon) = \infty(\epsilon)$ for $\delta > 0$. Define, for $u \in (\zeta)$ and $\delta > 0$,

$$E(u;\delta) = Pf.\,g_\delta(u;0). \tag{4.2}$$

THEOREM (4.3). *Let $q(\eta_1, \eta_2)$ be a real polynomial such that $q(0,0) \neq 0$. For $u \in (\zeta)$ and $\delta > 0$, define $E(u;\delta)$ as above. Then $E(u;\delta) = \infty(\delta)$. Furthermore, if we set $E(u) = Pf.E(u;0)$, then E, considered as a functional on the function space (ζ), is a tempered distribution.*

THEOREM (4.4). *For any real polynomial q in two variables, there exists a tempered distribution E satisfying the equation $Eq = 1$.*

The proof of Theorem (4.3) will be found in the last part of this paper. Theorem (4.4) is a consequence of Theorem (4.3) as follows. We may assume

that the coordinates η_1, η_2 have been chosen in such a way that $q(0,0) \neq 0$. Then let E be the tempered distribution defined in Theorem (4.3). It must be shown that $Eq = 1$, or, what is the same thing, that $w \in (\zeta)$ implies that

$$E(qw) = \int_{R^2} w \, d\eta.$$

From equation (4.1), we get that

$$g_\delta(qw;\epsilon) = \int_{G(\delta,\epsilon)} w(\eta)\, d\eta,$$

and hence

$$E(qw;\delta) = Pf.\, g_\delta(u;0) = \int_{G(\delta)} w(\eta)\, d\eta,$$

so that

$$E(qw) = Pf.\, E(qw;0) = \int_{R^2} w(\eta)\, d\eta.$$

This proves Theorem (4.4).

5. Several lemmas. We assume $q(\eta_1, \eta_2)$ to be a polynomial such that $q(0,0) \neq 0$ and define $H(\eta)$ by equation (2.1).

LEMMA (5.1). *Let $\eta = (\eta_1, \eta_2)$ be a real vector such that $q(\eta) \neq 0$ and $H(\eta) \neq 0$. Then*

$$[q(\eta)]^{-1} = -\sum_{k=1}^{\sigma} a_k(\eta, \lambda_k),$$

where $\lambda_1, \cdots, \lambda_\sigma$ are the distinct roots of the equation $q(\lambda\eta_1, \lambda\eta_2) = 0$ and where $a_k(\eta, \lambda_k)$ is the residue of the function $[q(\lambda\eta_1, \lambda\eta_2)(\lambda - 1)]^{-1}$ of the complex variable λ at the point $\lambda = \lambda_k$.

Proof. By Cauchy's integral formula,

$$[q(\eta)]^{-1} = \frac{1}{2\pi i} \int_C [q(\lambda\eta_1, \lambda\eta_2)(\lambda - 1)]^{-1} \, d\lambda,$$

where C is a positively oriented circle with center at $\lambda = 1$ and radius small enough that C contains none of the points $\lambda_1, \cdots, \lambda_\sigma$ in its interior or on its circumference. $H(\eta) \neq 0$ implies that $q(\lambda\eta_1, \lambda\eta_2)$ is not constant in λ, and hence

$$\frac{1}{2\pi i} \int_{|\lambda| = R} [q(\lambda\eta_1, \lambda\eta_2)(\lambda - 1)]^{-1} \, d\lambda = 0$$

when R is greater than the maximum of the numbers $|\lambda_1|, \cdots, |\lambda_\sigma|$. The lemma then follows from the theorem of residues.

Remark (5.2). The number σ of distinct roots of the equation $q(\lambda\eta_1, \lambda\eta_2) = 0$ is constant in the region of R^2 where $H \neq 0$. See section 2.

LEMMA (5.3). *If η is as in Lemma (5.1),*

$$a_k(\eta, \lambda_k) = P_k(\eta, \lambda_k)[D^{m_k}p(\eta, \lambda_k)(\lambda_k - 1)]^{-m_k}$$

where P_k is a polynomial in λ_k and η, where $D = d/d\lambda$, and where m_k is the multiplicity of λ_k considered as a zero of $q(\lambda\eta_1, \lambda\eta_2)$. Here $k = 1, \cdots, \sigma$.

This lemma is proved by finding the coefficient of $(\lambda - \lambda_k)^{-1}$ in the Laurent expansion of the function $[q(\lambda\eta_1, \lambda\eta_2)(\lambda - 1)]^{-1}$ about $\lambda = \lambda_k$. We omit the short calculation.

It is convenient to let Ω denote the unit circle in the $\eta_1\eta_2$-plane and to denote by ξ_1, ξ_2 respectively the restrictions of η_1, η_2 to Ω. Let the single-valued algebraic function $\lambda_k(\xi)$ be a branch of the complete algebraic function $\lambda(\xi)$ defined on $\Omega(\xi)$ by the equations $q(\lambda\xi_1, \lambda\xi_2) = 0$ and $\xi_1^2 + \xi_2^2 = 1$.

LEMMA (5.4). *If $\xi^0 \in \Omega$, let ξ_j, where $j = 1$ or 2, be a coordinate function for Ω in a neighborhood of ξ^0. Then there exists a positive integer h, a determination z of $(\xi_j - \xi^0_j)^{1/h}$ and a positive number τ such that $\lambda_k(\xi)$ can be expanded in a convergent power series*

$$\lambda_k(\xi) = \sum_{i=0}^{\infty} b_i z^i \tag{5.5}$$

for $|z| < \tau$.

Proof. See Picard [5], Vol. II, Chap. 13.

6. The function $J_{\alpha\beta}(u; \xi, \lambda)$. Let A denote the complex λ-plane with the non-negative real axis removed. For $\lambda \in A$, $u \in (\zeta)$, and $\xi \in \Omega$, set

$$J_{\alpha\beta}(u; \xi, \lambda) = \int_0^\infty t^\alpha u(t\xi_1, t\xi_2)(\lambda - t)^{-\beta}\, dt$$

where α, β are any non-negative integers. When λ is a non-negative real number, the integral on the right need not converge, so that we must define $J_{\alpha\beta}$ somewhat differently in this case. For λ real, $\lambda > 0$, set

$$g(\epsilon) = \int_0^{\lambda-\epsilon} t^\alpha u(t\xi_1, t\xi_2)(\lambda - t)^{-\beta}\, dt + \int_0^{\lambda-\epsilon} t^\alpha u(t\xi_1, t\xi_2)(\lambda - t)^{-\beta}\, dt,$$

where ϵ is a positive number small enough that $\lambda - \epsilon > 0$. A repeated integration by parts of the two integrals on the right side of the above

equation shows that $g(\epsilon)=\infty(\epsilon)$, so that the number $Pf.g(0)$ is well defined. Set

$$J_{\alpha\beta}(u;\xi,\lambda)=Pf.g(0)$$

for all real positive λ. We leave $J_{\alpha\beta}$ undefined for $\lambda=0$.

LEMMA (6.1). (i) *Restricted to positive real values of* λ, $J_{\alpha\beta}$ *is a* C^∞ *function of* λ, ξ_1 *and* ξ_2. (ii) $J_{\alpha\beta}$ *and all its partial derivatives with respect to* ξ_1 *and* ξ_2 *are regular analytic functions of* λ *for* $\lambda\in A$. (iii) *Let* a_0 *be a real number,* $a_0>0$, *and let* λ *approach the point* a_0 *in such a way that* $\operatorname{Im}\lambda$ *is non-zero and of constant sign. Then if* $P(d/d\lambda,\partial/\partial\xi)J_{\alpha\beta}$ *denotes an arbitrary partial derivative of* $J_{\alpha\beta}$ *with respect to* λ,ξ_1, *and* ξ_2, $P(d/d\lambda,\partial/\partial\xi)J_{\alpha\beta}$ *has a well-defined finite limit as* $\lambda\rightarrow a_0$.

Proof. (i) is verified by a straightforward calculation which we omit. (ii) follows at once from well-known theorems. We prove (iii). Since the expression $P(d/d\lambda,\partial/\partial\xi)J_{\alpha\beta}$ differs in no significant respect from $J_{\alpha\beta}$, it suffices to prove (iii) for the latter.

For $\lambda\in A$ and t in the interval $0\leqq t<\infty$, we have that $0<\arg(\lambda-t)<2\pi$. For λ and t in these ranges, we define $\operatorname{Log}(\lambda-t)=\log|\lambda-t|+i\arg(\lambda-t)$. Then for $\lambda\in A$, we integrate by parts and obtain that

$$J_{\alpha\beta}(u;\xi,\lambda)=\sum_{k=-\beta+1}^{-1}c_k\lambda^k+c_0\int_0^\infty\operatorname{Log}(\lambda-t)f(t\xi)\,dt+c_1\operatorname{Log}\lambda,$$

where the c_k are certain complex numbers and $f(t\xi)$ is the β-th derivative of $t^\alpha u(t\xi_1,t\xi_2)$ with respect to t. Furthermore, we can write

$$\text{(6.2)}\qquad \int_0^\infty\operatorname{Log}(\lambda-t)f(t\xi)\,dt$$
$$=\int_0^\infty\log|\lambda-t|\,f(t\xi)\,dt+i\int_0^\infty\arg(\lambda-t)f(t\xi)\,dt.$$

Let $\lambda=a+bi$ and suppose $\lambda\rightarrow a_0$ in such a way that b is always positive. Clearly, the first integral on the right side of (6.2) tends to the limit $\int_0^\infty\log|a_0-t|\,f(t\xi)\,dt$. If we define $h(t)=0$ for $0\leqq t<a_0$ and $h(t)=\pi$ for $a_0\leqq t<\infty$, then the second integral on the right side of (6.2) tends to the limiting value $i\int_0^\infty h(t)f(t\xi)\,dt$ as $\lambda\rightarrow a_0$, since $\arg(\lambda-t)$ is bounded and converges uniformly to $h(t)$ as $|a-a_0|+b\rightarrow 0$ in every subset of the real line which excludes a neighborhood of the point a_0. This concludes the proof of (iii) and hence of the lemma.

Let $\xi^0 \in \Omega$ and let z be the function defined in Lemma (5.4). Then $z = \phi w$ where ϕ is an h-th root of unity and w is a real parameter lying in the interval $-\tau < w < \tau$.

LEMMA (6.3). (i) *There exists a positive number τ such that $J_{\alpha\beta}(u;\xi,\lambda_k(\xi))$ is infinitely differentiable with respect to w for $0 < w < \tau$ and for $-\tau < w < 0$. Furthermore, all derivatives of $J_{\alpha\beta}(u;\xi,\lambda_k(\xi))$ with respect to w are continuous in the half-closed intervals $0 \leqq w < \tau$ and $-\tau < w \leqq 0$.* (ii) *$J_{\alpha\beta}$ has the Taylor expansion*

$$J_{\alpha\beta}(u;\xi,\lambda_k(\xi)) = \sum_{j=0}^{n} d_j w^j + w^{n+1} K_n(w) \tag{6.4}$$

for $0 \leqq w < \tau$, where n is any positive integer, the d_j are complex numbers, $K_n(w)$ is infinitely differentiable for $0 < w < \tau$ and all its derivatives are continuous in the half-closed interval $0 \leqq w < \tau$. (iii) *An expansion similar to (6.4) exists in the interval $0 \geqq w > -\tau$.*

Proof. (ii) and (iii) follow immediately from (i), so we prove only the latter. For the sake of brevity, we restrict ourselves to showing the existence of an interval $0 < w < \tau$ where (i) is true; a repetition of the argument will prove (i) for small negative values of w. We choose τ small enough that $\lambda_k(\xi(w))$ is either everywhere real or everywhere non-real in the interval $0 < w < \tau$. If λ_k is real for $0 \leqq w < \tau$, then (i) follows from Lemmas (5.4) and (6.1) since a composition of differentiabie functions is differentiable. The same argument applies in the other two possible cases as well, except that if λ_k is real for $w = 0$ but non-real for $w \neq 0$, the derivatives of $J_{\alpha\beta}$ may have a jump discontinuity at $w = 0$. But in any case, they have well-defined finite limits as $w \to 0$. This proves Lemma (6.3).

Let $\Omega(\delta)$, for small positive δ, denote the intersection of the region $G(\delta)$ with the unit circle Ω and for small positive ϵ let $S(\epsilon,\theta)$ denote that part of the interval $0 \leqq t < \infty$ where $|q(t\cos\theta, t\sin\theta)| > \epsilon$. Then for $\epsilon > 0$ and θ such that $(\cos\theta, \sin\theta) \in \Omega(\delta)$, set

$$w(\epsilon,\theta) = \int_{S(\epsilon,\theta)} t^{\alpha} u(t\cos\theta, t\sin\theta)(\lambda_k(\theta) - t)^{-\beta}\, dt, \tag{6.5}$$

where $\lambda(\theta)$ is a solution of the equation $q(\lambda_k(\theta)\cos\theta, \lambda_k(\theta)\sin\theta) = 0$ and α, β are arbitrary non-negative integers.

LEMMA (6.6). *Let $C(\delta)$ be the closure of some component of $\Omega(\delta)$. Then there exists a positive number ϵ_0, a finite set M of negative fractions, functions $c_\mu(\theta)$, where μ runs over M, and functions $c(\theta)$, $d(\theta,\epsilon)$ such that:* (i)

$c_\mu(\theta)$, $c(\theta)$ are defined and continuous for all θ such that $(\cos\theta, \sin\theta) \in C(\delta)$; (ii) *$d(\theta, \epsilon)$ is defined and continuous in both θ and ϵ for θ such that* $(\cos\theta, \sin\theta) \in C(\delta)$ *and* $0 \leqq \epsilon < \epsilon_0$; (iii) $w(\epsilon, \theta) = \sum_{\mu \in M} c_\mu(\theta)\epsilon^\mu + c(\theta)\log\epsilon + d(\theta, \epsilon)$ *for all such values of ϵ and θ.*

Proof. Since by assumption $q(0,0) \neq 0$, we can choose a positive number γ_0 such that $q(0,0) \neq \tau$ whenever τ is a real number satisfying the inequality $|\tau| < \gamma_0$. For real τ, let

$$q(\eta_1, \eta_2) - \tau = \prod_{i=1}^{\nu} [q_i(\eta_1, \eta_2; \tau)]^{\alpha_i}$$

be a decomposition of $q(\eta_1, \eta_2) - \tau$, considered as a polynomial in η_1 and η_2, into integral powers α_i of irreducible polynomials $q_i(\eta_1, \eta_2; \tau)$ and let $D_j(\eta_1, \eta_2; \tau)$ be the discriminant of the polynomial $q_j(\lambda\eta_1, \lambda\eta_2; \tau)$, $R_{ij}(\eta_1, \eta_2; \tau)$ the resultant of the polynomials $q_i(\lambda\eta_1, \lambda\eta_2; \tau)$ and $q_j(\lambda\eta_1, \lambda\eta_2; \tau)$ where these latter are considered as polynomials in λ alone. Then set

$$H(\eta_1, \eta_2; \tau) = \prod_{\substack{i,j=1 \\ i<j}}^{\nu} R_{ij}(\eta_1, \eta_2; \tau) D_j(\eta_1, \eta_2; \tau).$$

$H(\eta_1, \eta_2; \tau)$ is homogeneous in η_1, η_2, so that $H = 0$ is a cone. By the definition of H, a line through the origin belongs to this cone if it is tangent to or passes through an exceptional point of the curve $q = \tau$.

Let $\Gamma(\delta)$ denote the angular sector of R^2 consisting of all points which lie on a line passing through both the origin and a point of $C(\delta)$. We have that $H(\eta_1, \eta_2; 0)$ vanishes in $\Gamma(\delta)$ at just the point $\eta_1 = \eta_2 = 0$. Since $H(\eta_1, \eta_2; \tau)$ is a continuous function of τ and homogeneous in η_1 and η_2 we can choose a positive number γ_1 less than γ_0 and such that for $|\tau| < \gamma_1$ the polynomial $H(\eta_1, \eta_2; \tau)$ does not vanish in $\Gamma(\delta)$ except at the origin.

Now let V_τ denote the curve $q = \tau$ and consider that part of it lying in $\Gamma(\delta)$. By our construction, if $|\tau| < \gamma_1$, $V_\tau \cap \Gamma(\delta)$ will consist of a number of disjoint components $V_\tau^1, \cdots, V_\tau^s$, each of which is non-singular and nowhere tangent to a line passing through the origin. If ρ denotes the radial distance from the origin, V_τ^i will be given by the equation $\rho = \rho^i(\tau, \theta)$ where ρ^i is an analytic function of τ and θ for $|\tau| < \gamma_1$, θ such that $(\cos\theta, \sin\theta) \in C(\delta)$. Furthermore, since the components $V_\tau^1, \cdots, V_\tau^s$ are disjoint, we may assume that $0 < \rho^1(\tau, \theta) < \cdots < \rho^s(\tau, \theta)$. From the definition of $\lambda_k(\theta)$, we see that there are two possible cases: (i) $\lambda_k(\theta) \neq \rho^i(0, \theta)$ for $i = 1, \cdots, s$ and all θ such that $(\cos\theta, \sin\theta) \in C(\delta)$; (ii) there exists a unique integer j, $1 \leqq j \leqq s$, such that $\lambda_k(\theta) \equiv \rho^j(0, \theta)$ while $\lambda_k(\theta) \neq \rho^i(0, \theta)$

for $i=1,2,\cdots,j-1,j+1,\cdots,s$ and all θ such that $(\cos\theta,\sin\theta)\in C(\delta)$.

In the first case, there is nothing to prove since $w(\epsilon,\theta)$ will then be continuous at $\epsilon=0$. So assume (ii) holds. Then $V_0{}^j$ will be flanked by two curve segments, one from the family $V_\epsilon{}^i$ and one from the family $V_{-\epsilon}{}^i$. It is convenient to choose our notation in such a way that $V_{-\epsilon}{}^i$ is given by the equation $\rho=\rho(-\epsilon,\theta)$, $V_0{}^j$ by the equation $\rho=\rho(0,\theta)$, and $V_\epsilon{}^i$ by the equation $p=p(\epsilon,\theta)$. Without loss of generality, we can assume that $\rho(-\epsilon,\theta)<\rho(0,\theta)<\rho(\epsilon,\theta)$ for ϵ positive, $\epsilon<\gamma_2<\gamma_1$, and for θ such that $(\cos\theta,\sin\theta)\in C(\delta)$. This being so, we have that

$$(6.7)\qquad w(\epsilon,\theta)=\int_0^{\rho(-\epsilon,\theta)} t^\alpha u(t\cos\theta,t\sin\theta)(\lambda_k(\theta)-t)^{-\beta}\,dt + \int_{\rho(\epsilon,\theta)}^{\infty} t^\alpha u(t\cos\theta,t\sin\theta)(\lambda_k(\theta)-t)^{-\beta}\,dt + W(\epsilon,\theta),$$

where $W(\epsilon,\theta)$ is continuous for $0\leqq\epsilon<\gamma_2$ and θ such that $(\cos\theta,\sin\theta)\in C(\delta)$ and where $W(0,\theta)\equiv 0$. It therefore suffices to prove that the integrals appearing on the right side of equation (6.7) have an expansion of the form required by (iii) of the lemma. We consider only the first integral and denote it by $w_1(\epsilon,\theta)$; the second integral can be treated in an analogous manner. Setting $f(t,\theta)\equiv t^\alpha u(t\cos\theta,t\sin\theta)$ and integrating by parts β times, we obtain

$$(6.8)\qquad \begin{aligned} w_1(\epsilon,\theta)=&\sum_{i=1}^{\beta-1} b_i f^{(i-1)}(\rho(-\epsilon,\theta),\theta)(\lambda_k(\theta)-\rho(-\epsilon,\theta))^{i-\beta}\\ &+b_0 f^{(\beta-1)}(\rho(-\epsilon,\theta)\log|\lambda_k(\theta)-\rho(-\epsilon,\theta)|\\ &+b\int_0^{\rho(-\epsilon,\theta)} f^{(\beta)}(\rho(-\epsilon,\theta)\log|\lambda_k(\theta)-\rho(-\epsilon,\theta)|\,dt\\ &+\cdots, \end{aligned}$$

where $+\cdots$ indicates that we have omitted those boundary terms (independent of ϵ) which arise from the limit of integration $t=0$, where b_i, b_0, b are constants and where $f^{(i)}$ denotes the i-th partial derivative of f with respect to t.

Now the function $\rho(\epsilon,\theta)$ satisfies the equation

$$q_i(\rho(\epsilon,\theta)\cos\theta,\rho(\epsilon,\theta)\sin\theta)=\epsilon$$

for some irreducible polynomial q_i and hence

$$\partial\rho/\partial\epsilon=[(\partial q_i/\partial\eta_1)\cos\theta+(\partial q_i/\partial\eta_2)\sin\theta]^{-1}.$$

Therefore $|\partial\rho/\partial\epsilon|\geqq|\operatorname{grad} q_i|^{-1}>0$. It follows that we can write

$$[\lambda_k(\theta)-\rho(-\epsilon,\theta)]^{-1}\equiv\epsilon^{-1}\pi(\epsilon,\theta)$$

for $\epsilon < \epsilon_0 < \gamma_2$ and for all θ such that $(\cos\theta, \sin\theta) \in C(\delta)$ and where $\pi(\epsilon,\theta)$ is analytic for these values of ϵ and θ.

For such values of ϵ and θ, the integral on the right side of equation (6.8) is obviously continuous. Consider the remaining terms on the right side of this equation. Each has an expansion of the kind required by (iii) of the lemma, since $f^{(i-1)}(\rho(-\epsilon,\theta),\theta)$ is infinitely differentiable in ϵ and can be expanded in a Taylor's expansion in ϵ about $\epsilon = 0$, while by what we have said above, the quantity $[\lambda_k(\theta) - \rho(-\epsilon,\theta)]^{i-\beta}$ can be expanded in a series of negative and positive powers of ϵ and the coefficients of both expansions will be continuous functions of θ as desired. Therefore $w_1(\epsilon,\theta)$ has an expansion of the form (iii). This proves Lemma (6.6).

We recall that for $u \in (\zeta)$ and for $\delta > 0$, the number $E(u;\delta)$ is defined by equation (4.2).

LEMMA (6.9). *$E(u,\delta)$ is equal to a linear combination of terms of the form*

$$\int_{\Omega(\delta)} R(\theta) J_{\alpha\beta}(u;\xi(\theta),\lambda_k(\theta))\,d\theta, \tag{6.10}$$

where α, β are non-negative integers and for some other non-negative integers i, j_1, j_2,

$$R(\theta) = [\lambda_k(\theta)]^i[\xi_1(\theta)]^{j_1}[\xi_2(\theta)]^{j_2}[D^{m_k}p(\xi(\theta),\lambda_k(\theta))]^{-m_k}$$

where $D = (d/d\lambda)$ and where m_k is the multiplicity of $\lambda_k(\theta)$ considered as a root of the equation $p(\xi(\theta),\lambda_k(\theta)) = 0$.

Proof. By definition (equation (4.1)), $g_\delta(u;\epsilon) = \int_{G(\delta,\epsilon)} u(\eta)[q(\eta)]^{-1}\,d\eta$. Then by Lemmas (5.1) and (5.3), $g_\delta(u;\epsilon)$ is a linear combination of expressions of the form:

$$\int_{G(\delta,\epsilon)} u(\eta)\eta_1^{j_1}\eta_2^{j_2}[\lambda_k(\eta)]^i[D^{m_k}p(\eta,\lambda_k(\eta))(\lambda_k(\eta)-1)]^{-m_k}\,d\eta. \tag{6.11}$$

Introducing polar coordinates $\eta_1 = t\cos\theta$, $\eta_2 = t\sin\theta$, and using the formula $\lambda_k(\eta_1,\eta_2) = t^{-1}\lambda_k(\cos\theta,\sin\theta) = t^{-1}\lambda_k(\theta)$, we find that

$$D^{m_k}p(\eta,\lambda_k(\eta)) = t^{m_k}D^{m_k}p(\cos\theta,\sin\theta,\lambda_k(\theta)).$$

Hence (6.11) becomes

$$\int_{\Omega(\delta)} R(\theta)\,d\theta \int_{S(\epsilon,\theta)} t^{\alpha}u(t\cos\theta,t\sin\theta)(\lambda_k(\theta)-t)^{-m_k}\,dt, \tag{6.12}$$

where $\alpha = j_1 + j_2 - i + m_k(1-m_k) + 1$ and where $S(\epsilon,\theta)$ is that part of the interval $0 \leqq t < \infty$ where $|q(t\cos\theta,t\sin\theta)| > \epsilon$.

Keeping δ fixed always, let $h(\epsilon)$ denote the expression (6.12). To prove Lemma (6.9), we must show that $h(\epsilon) = \infty(\epsilon)$ and further that

$$Pf.h(0)=\int_{\Omega(\delta)} R(\theta)J_{\alpha\beta}(u;\xi(\theta),\lambda_k(\theta))\,d\theta,$$

where we have put $\beta=m_k$. Define $w(\epsilon,\theta)$ as in equation (6.5). Then, by Lemma (6.6), we can write $w(\epsilon,\theta)=\sum_{\mu\in M} c_\mu(\theta)\epsilon^\mu+c(\theta)\log\epsilon+d(\theta,\epsilon)$. Then

$$h(\epsilon)=\sum_{\mu\in M}\epsilon^\mu\int_{\Omega(\delta)} R(\theta)c_\mu(\theta)\,d\theta+\log\epsilon\int_{\Omega(\delta)} R(\theta)c(\theta)\,d\theta$$
$$+\int_{\Omega(\delta)} R(\theta)d(\theta,\epsilon)\,d\theta.$$

From this equation, it follows that

$$Pf.h(0)=Pf.\{\int_{\Omega(\delta)} R(\theta)d(\theta,\epsilon)\,d\theta\}_{\epsilon=0},$$

provided the quantity in brackets on the right has a finite limit as $\epsilon\to 0$. However, this is the case since, by Lemma (6.6), $d(\theta,\epsilon)$ is continuous, and therefore

$$\lim_{\epsilon\to 0}\int_{\Omega(\delta)} R(\theta)d(\theta,\epsilon)\,d\theta=\int_{\Omega(\delta)} R(\theta)[\lim_{\epsilon\to 0} d(\theta,\epsilon)]\,d\theta.$$

This proves Lemma (6.9).

7. **Proof of Theorem (4.3).** This theorem states that $E(u;\delta)=\infty(\delta)$. By Lemma (6.9), it suffices to show that an expression of the form (6.10) is $\infty(\delta)$ and this in turn amounts to showing that

$$\int_{\theta^0+\delta}^{\theta^0+\tau} R(\theta)J_{\alpha\beta}(u;\xi(\theta),\lambda_k(\theta))\,d\theta=\infty(\delta), \tag{7.1}$$

where θ^0 is a solution of the equation $H(\cos\theta^0,\sin\theta^0)=0$ and where τ is a positive number such that $\tau>\delta$ and so small that the interval $\theta^0<\theta<\theta^0+\tau$ contains no solution of this equation. Let $\xi_1{}^0=\cos\theta^0$, $\xi_2{}^0=\sin\theta^0$ and define the parameter w on Ω as in Lemma (6.3). By equation (5.5), equation (6.4), and the definition of $R(\theta)$, we can write

$$R(\theta)J_{\alpha\beta}(u;\xi(\theta),\lambda_k(\theta))\,d\theta=\{\sum_{\mu\in M}\gamma_\mu w^\mu+\gamma\log|w|+c(w)\}\,dw, \tag{7.2}$$

where γ_μ, γ are numbers independent of w, M is a finite set of negative fractions, and $c(w)$ is continuous for those values of w corresponding to the interval $\theta^0\leqq\theta<\theta^0+\delta$. Substituting the expansion (7.2) inside the interval on the left side of equation (7.1) and integrating term by term, we prove equation (7.1) and hence that $E(u;\delta)=\infty(\delta)$.

The proof of Theorem (4.3) will be complete if we show that, considered as a functional on the space (ζ), E is a tempered distribution. It is obvious

that E is linear, so we need only establish its continuity. We consider a sequence $\{u_n\}$ where u_n is a C^∞ function of η_1, η_2 which tends to zero in the sense that for an arbitrary polynomial $P(\eta_1,\eta_2)$ and an arbitrary but fixed partial derivative D, the sequence $\{PDu_n\}$ tends to zero uniformly in R^2. To show the continuity of E, it suffices to show that $E(u_n)\to 0$ as $n\to\infty$.

Under these assumptions concerning the sequence $\{u_n\}$, we shall have that $J_{\alpha\beta}(u_n;\xi,\lambda)\to 0$ for all fixed ξ and λ and the convergence is uniform for $\xi\in\Omega$, $\lambda\neq 0$. This follows from the definition of $J_{\alpha\beta}$. Moreover, the statement remains true if we replace $J_{\alpha\beta}$ by $D_1J_{\alpha\beta}$, where D_1 represents any partial differentiation with respect to ξ and λ.

By Lemma (6.9), we shall be finished if we verify that the expression

$$\int_{\Omega(\delta)} R(\theta)J_{\alpha\beta}(u_n;\xi(\theta),\lambda_k(\theta))\,d\theta \tag{7.3}$$

tends to zero as $n\to\infty$. But by formula (7.2), we have that

$$R(\theta)J_{\alpha\beta}(u_n;\xi(\theta),\lambda_k(\theta))\,d\theta=\{\sum_{\mu\in M}\gamma_\mu w^\mu+\gamma\log|w|+c(w)\}dw.$$

Since the γ_μ, γ all contain a derivative of $J_{\alpha\beta}$ as a factor, they tend to zero as $n\to\infty$. The function $c(w)$ contains as a factor the remainder term of a Taylor's expansion of $J_{\alpha\beta}$ about $w=0$. Since all derivatives of $J_{\alpha\beta}$ converge uniformly to zero as $n\to\infty$, it follows that the integral of $c(w)$ will tend to zero as $n\to\infty$. Hence (7.3) tends to zero as $n\to\infty$. This proves that E is continuous, and therefore Theorem (4.3) is established.

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LINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS.*

By JAMES C. LILLO.[1]

Introduction. We shall be interested in linear systems of the form $x' = A(t)x + B(t)$, where $A(t)$ is an $n \times n$ matrix, $B(t)$ is an n vector, and their entries are almost periodic functions (either real or complex valued). Historically, the approach to this type of problem has been to assume that the solutions of the system $x' = A(t)x$ have certain properties and then obtain the existence of an almost periodic solution of $x' = A(t)x + B(t)$. The limitations of such a procedure are apparent. While the results obtained in this paper are restricted to rather special types of almost periodic matrices, they do possess the advantage that the restrictions are placed directly upon the coefficients. For the case $B \equiv 0$, we shall concern ourselves with a modified form of the representation problem considered by Cameron [4]. Our results divide into two main types, the first type being when $A(t)$ is a superdiagonal matrix. An example (example A) is obtained which illustrates the difficulties involved and suggests the restrictions imposed in Theorem 1. The second type of matrix $A(t)$ considered is that in which the frequencies of the $a_{ij}(t)$ are all positive and bounded away from zero. The results in this case are an extension of earlier results due to Wintner and Putnam [12]. For the case $B \not\equiv 0$, we consider the existence of almost periodic solutions. The main result is found in Theorem 3. We shall employ the notation used by Besicovitich [1], writing a. p. for almost periodic and denoting the unique association of an a. p. function with its Fourier series by $\sim$.

Part I. We consider here systems of the form 1) $x' = A(t)x$, where $A(t)$ is an $n \times n$ matrix of a. p. functions. A classical question for systems of this type is when may the fundamental solution be written in the form $\phi(t) = P(t)\exp(At)$, where $P(t)$ is an a. p. matrix and A is a constant

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matrix. We consider here a slightly more general representation problem. We ask when the components of every solution of 1) may be written in the form $\exp(\lambda_j) \sum_{l=0}^{k} t^l/l!\, P_l(t)$, where $P_l(t)$ is a. p. and k is equal to or less than the multiplicity of λ_j. We consider first the case in which the matrix $A(t)$ is superdiagonal. The results of Perron [10] and Diliberto [5] assure us that any equation $x' = D(t)x$, where $D(t)$ is bounded and continuous, is kinematically [8] similar to an equation of the form $x' = A(t)x$, where $A(t)$ is superdiagonal. Unfortunately, it is not known under what conditions the resulting $A(t)$ will be almost periodic when the $B(t)$ is so restricted. Thus for the present at least, the restriction to the superdiagonal is a serious limitation. The results obtained, however, are quite complete, and example A in particular has rather interesting implications. In preparation for the study of superdiagonal systems, we have the following three lemmas.

LEMMA 1. *If* $p(t) \sim \sum_{j=1}^{\infty} a_j \exp(i\Lambda_j t)$ *is an* a. p. *function and* $\operatorname{Re}\lambda > 0$, *then*

$$q(t) = \int_t^\infty p(s)\exp[\lambda(t-s)]\,ds \text{ is a. p. and } q(t) \sim \sum_{j=1}^{\infty} a_j \exp(i\Lambda_j t)/(\lambda + i\Lambda_j).$$

Proof. The a. p. nature of $q(t)$ was established by Murray [9]. The nature of the Fourier development of $q(t)$ may be obtained by a simple generalization of the technique used by Bohr [3] to show the analogous fact for the integral of a. p. functions.

LEMMA 1′. *If* $p(t) \sim \sum_{h=1}^{\infty} a_h \exp(i\Lambda_h t)$ *is an* a. p. *function and* $\operatorname{Re}\lambda > 0$, *then* $\int_t^\infty p(s)[(t-s)^m t^l \exp[\lambda(t-s)]]/(m!\,l!)\,ds = \sum_{j=0}^{l} (t^j/j!)q^j(t)$, *where each* $q^j(t)$ *is* a. p., $q^j(t) \sim \sum_{h=1}^{\infty} [a_h/(\lambda + i\Lambda_h)^{m+1+j}]\exp(i\Lambda_h t)$. *Furthermore, if* $M = \text{l. u. b.}\,|p(t)|$, *the* l. u. b. $|q^j(t)| \leqq 2M/\lambda^{m+1}$.

Proof. The result is established in essentially the same way as Lemma 1.

LEMMA 2. *Given any* a. p. *function* $b(t) \sim \sum_{j=1}^{\infty} b_j \exp(i\Lambda_j t) + b_0$, *where* $|\Lambda_j - \Lambda_i| > \delta > 0$ $(j \neq i)$, $|\Lambda_i| > \delta > 0$, *there exists an* a. p. *function* $h(t) \sim \sum_{l=1}^{\infty} h_l \exp(+\gamma_l t)$, *where* $|\gamma_l - \gamma_{l+1}| > \delta/2$ *and* $|\gamma_l| > \delta/2 > 0, i, l = 1, 2, \cdots$, *such that* $\int_0^t b(s)h(s)\,ds = f(t)$ *is not an* a. p. *function.*

Proof. Define $\gamma_j = [-\Lambda_j + b_j h_j/(j+k)^2]$ where h_j are chosen so that $b_j h_j \neq b_k h_k$ $(j \neq k)$, $\sum_{l=0}^{\infty} |h_l|^2$ is bounded, and k is a positive integer such that $\mathrm{l.u.b.}\, |b_j h_j|/k^2 < \delta/4$. Then $h(t)$ is an a.p. function and its indefinite integral is also an a.p. function. We now establish the fact that the integral of $b(t)h(t)$ is not a.p. Now

$$b(t)h(t) \sim [\sum_{j=1}^{\infty} b_j \exp(i\Lambda_j t) + b_0][\sum_{l=1}^{\infty} h_l \exp(-i\gamma_l t)],$$

and so, if the indefinite integral of $b(t)h(t)$ were an a.p. function, by known results, its Fourier development would have to be similar to that obtained by formally integrating the above formal product. However, if one formally does this, he obtains a Fourier series $\sum_{l,j=1}^{\infty} c_{lj} \exp[(i\Lambda_j \gamma_l)t]$ for which the partial sums $\sum_{l,j=1}^{\infty} (|c_{lj}|)^2$ are unbounded. Thus we may conclude that the integral of the product $b(t)h(t)$ is not an a.p. function.

We here note that it is possible to restrict oneself to real a.p. functions and still obtain the results of Lemma 2. The necessary modifications are suggested by the trigonometric identity: $(\cos\Lambda_n + \sin\Lambda_n)(\cos\gamma_n - \sin\gamma_n) = \cos(\Lambda_n + \gamma_n) + \sin(\Lambda_n - \gamma_n)$. In this connection, it is noted that the example which appears in the following paragraph may be similarly modified.

Consider now the linear system of the form $x' = A(t)x$, where $A(t)$ is a.p. and superdiagonal. We begin by using the results obtained in Lemma 2, to construct an example which illustrates the difficulties involved even for very special systems. Upon considering this example, the hypotheses of Theorem 1 and its corollary are immediately suggested.

Example A. We consider the linear system in two variables of the form:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a_1(t) & a_2(t) \\ 0 & b_1(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where the functions $a_1(t)$ and $b_1(t)$ are periodic 2π and mean M. Therefore $\int_0^t a_1(s)\,ds = Mt + h(t)$ and $\int_0^t b_1(s)\,ds = Mt + c(t)$, where $c(t)$ and $h(t)$ are periodic with zero mean. We have that $(\exp(Mt + h(t)), 0)$ and $(\exp(Mt + h(t)) \cdot [1 + \int_0^t \exp(-h(s) + c(s)) \cdot a(s)\,ds], \exp(Mt + c(t)))$ form a basis for the solution space. Since $\exp[-h(t) + c(t)]$ is periodic, it clearly satisfies the conditions of Lemma 2 for $b(t)$ so that if we define $a_2(t)$ as the corresponding $h(t)$ of Lemma 2, then the given system will not admit a representation of the desired form.

Since, upon repeated integration of the functions $a_1(t) - M$, $b_1(t) - M$, and $a_2(t)$, we always obtain an a. p. function, it is clear that the imposing of integrability conditions on the $a_{ij}(t)$ will not suffice to give a representation result. A spacing condition on the frequencies is likewise seen to be adequate. The results of Lemma 1 suggest, of course, that a restriction be put on the integrability and mean values of the diagonal terms. This is done in the following theorem.

THEOREM 1. *If the system* $x' = A(t)x = (a_{ij}(t))x$ *satisfies* 1) $a_{ij}(t) \equiv 0$ *for* $i < j$, 2) *the* $a_{ij}(t)$ *are* a. p., 3) $\int_0^t a_{ii}(s)\,ds = m_i t + p_i(t)$, *where* $p_i(t)$ *is* a. p. *and* $m_i \neq m_j$ *for* $i \neq j$, *then there exists a fundamental set of solutions* $x_1(t), \cdots, x_n(t)$, *where* $x_h{}^j(t) = \exp(m_h t) \sum_{i=1}^{h} P_{hj}{}^i(t)$, $j \leqq h$; $x_h{}^j(t) = 0$, $j > h$. *Here the* $P_{hj}{}^i(t)$ *are* a. p. *with their module contained in that of* $A(t)$.

Proof. This result is an immediate consequence of repeated applications of Lemma 1.

COROLLARY. *If, in addition to the hypothesis of Theorem* 1, *we insist that the* m_i *be lineary ordered and that no solution* $x(t)$ *having an exponential multiplier* $\exp(\lambda t)$ *be such that* $\exp(-\lambda t)x(t)$ *is bounded and positive twistable with respect to the module of* A, *then any fundamental solution* $\Phi(t)$ *of* $x' = A(t)x$ *has a representation of the form* $\Phi(t) = P(t)\exp(At)$, *where* A *is a constant matrix and* $P(t)$ *is an* a. p. *matrix whose module is contained in the module of* $A(t)$.

Proof. The proof is a straightforward application of Theorem XIII of Cameron's paper [4].

We next consider linear systems of the form 1) where the $a_{ij}(t)$ are a. p. functions whose frequencies are all positive (or all negative) and bounded away from zero. This type of system was first considered by Putnam and Wintner [12] in the special case of a second order equation. Their results were extended by S. Sandor [11] to the case of an n-th order equation. In Theorem 2, we extend these results to general n-dimensional linear systems.

Thus we consider the system $x' = A(t)x$, where $A(t) = (a_{ij}(t))$ and the $a_{ij}(t)$ are a. p. functions having frequencies which are all of the same sign, say positive, and bounded away from zero. Let Λ denote the g. l. b. of the absolute values of the frequencies of the $a_{ij}(t)$. We may assume, without loss of generality, that $\Lambda > 2$ (i. e. a change of parameter will make it such). Let A_0 denote the matrix formed by the constant terms of the $a_{ij}(t)$. We insist that the imaginary parts of the roots λ_j of A_0 are less than Λ so, as above, we may assume $\gamma = \min(\Lambda\text{-}\mathrm{Im}\,\lambda) > 2$.

THEOREM 2. *Corresponding to each solution of* $x' = A_0 x$ *of the form* $\exp(\lambda_i t)[0,\cdots,0,t^m/m!,0,\cdots,0]$, *the system* $x' = A(t)x$, *as defined above, has a solution each of whose components are of the form* $\exp(\lambda_i t)\sum_{l=0}^{m} t^l/l!\,p_l(t)$, *where the* $p_l(t)$ *are* a. p. *functions whose module is contained in the module of* $(A(t), \operatorname{Im}\lambda_j)$.

Proof. In this proof, we assume the frequencies are all positive since, by a change of parameter, this is always possible. Let $a(t)$ denote a general $a_{ij}(t)$. We shall be interested only in general properties of the $a_{ij}(t) \sim \sum_{l=1}^{m} c_l \exp(i\Lambda_l t)$. Thus for $\int_t^\infty \exp[\delta(t-s)]\cdot a_{ij}(s)\,ds \sim \sum_{l=1}^{\infty} c_l \exp(i\Lambda_l t)/(\delta + i\Lambda_l)$, we simply write $a(t)/\Lambda$. Similarly, for $\int_t^\infty \exp(t-s))\cdot a(s)\cdot a(s)/\Lambda\,ds$, we simply write $a^2(t)/(2!\Lambda^2)$, where the frequencies of $a^2(t)$ all exceed 2Λ. In the case of simple roots, we have, corresponding to the solution $x = \exp(\lambda_h t)[0,\cdots,0,1,0,\cdots,0]$ of the system $x' = A_0 x$, the solution $x(t) = x^0(t) + \sum_{l=1}^{\infty} x^l(t)$ of the system $x' = A(t)x$. Here $x^0(t) = \exp(\lambda_h t)[0,\cdots,1,0,\cdots,0]$, and the components of $x^l(t)$ are defined in terms of the components of $x^{l-1}(t)$ as follows for $j \neq h$: $x^l{}_j(t) = \exp(\lambda_j t)\int \exp(-\lambda_j s)\sum_{m=1}^{n} a_{mj}(t) x^{l-1}{}_m(s)\,ds$. If the real part of $(-\lambda_j + \lambda_h)$ is < 0 $[> 0]$, then the limits of integration in the above definition are taken as $\int_\infty^t$ $[\int_{-\infty}^t]$. For $x^l{}_h(t)$, we have

$$x^l{}_h(t) = \exp(\lambda_h t)[c^l + \int_0^t \exp(-\lambda_h s)\sum_{m=1}^{n} a_{im}(s) x^{l-1}{}_m(s)\,ds],$$

where c^l is the value of the integral at its lower limit of integration. Thus $x^1{}_j(t) = [a(t)/(\lambda_h - \lambda_j + i\Lambda)]\exp(\lambda_h t)$ and $x^1{}_h(t) = [a(t)/(i\Lambda)]\exp(\lambda_h t)$. In what follows, for convenience, we drop the $\lambda_h - \lambda_j + i\Lambda$ and simply write γ. Now assume $x^q{}_j(t) = n^{q-1} a^q(t)\exp(\lambda_h t)/(q!\gamma^q)$, where $a^q(t)$ is an a. p. function with no constant term and whose least frequency is greater than $q\gamma$. Then:

$$\begin{aligned} x^{q+1}{}_j(t) &= \exp(\lambda_j t)\int \exp(-\lambda_j s)\cdot\sum_{m=1}^{n} a_{jm}(s) x^q{}_m(s)\,ds \\ &= \exp(\lambda_h t) n^q a^{q+1}(t)/[(q+1)!\gamma^{q+1}], \qquad j \neq h, \end{aligned}$$

$$\begin{aligned} x^{q+1}{}_h(t) &= \exp(\lambda_h t)[c^{q+1} + \int_0^t \exp(-\lambda_h s)\sum_{l=1}^{n} a_{lh}(s) x^q{}_l(s)\,ds] \\ &= n^q \exp(\lambda_h t) a^{q+1}(t)/[(q+1)!\gamma^{q+1}]. \end{aligned}$$

In the case of $x^q{}_j(t)$, $j \neq h$, the limits of integration are established as previously stated. Here $a^{q+1}(t)$ is an a. p. function which again has no constant term and whose least frequency is greater than $(q+1)\gamma$. Since the $a_{ij}(t)$ are all a. p., there exists an M such that for all values of i, j, and t, $|a_{ij}(t)| < M$, from which it follows that $|a^{q+1}(t)| < (2M)^{q+1}/[(q+1)!\gamma^{q+1}]$. From this last inequality, we obtain $| x^{q+1}{}_h(t) \leqq \exp(\lambda_h t)[(2nM/\gamma)^{q+1}/(q+1)!]$. Thus for $x_h(t) = \sum_{q=0}^{\infty} x^q{}_h(t)$, we have the majorant $\exp(\lambda_h t + 2nM/\gamma)$, and for the series of a. p. functions which are multiplied by $\exp(\lambda_h t)$, we have the majorant $\exp(2nM/\gamma)$. By standard arguments, $x(t) = \sum_{q=0}^{\infty} x^q(t)$ is a solution of our equation. Since the uniform limit of a sequence of a. p. functions is again an a. p. function, the desired result follows in the case of simple roots. We next consider the case of multiple roots. For a solution corresponding to a given λ_k, of multiplicity $m+1$, the h-th component, corresponding to a root, $\lambda_j \neq \lambda_k$, has the basic integral

$$\exp(\lambda_j t)\int \exp(-\lambda_j s)s^m/m!a^q(s)\exp(\lambda_k s)(t-s)^h/h!a(s)\,ds.$$

Here we have $h \leqq r$, the maximum multiplicity of all the λ_j. Then integrating by parts, it follows that

$$\int_{\pm\infty}^{t} \exp[\lambda_j(t-s)](t-s)^h/h!a^q(s)s^m/m!\exp(\lambda_k s)a(s)\,ds$$
$$= \sum_{s=0}^{m} t^{m-s}a^{q+1}(t)\exp(\lambda_k t)/[(m-s)!(q\gamma)^{s+h+1}].$$

Thus, except for the fact that λ_j is a multiple root, so that now we may have up to r^2 terms and so that $\exp(2nMr^2/(\gamma)+\lambda_k t)$ is used instead of $\exp(2nM/(\gamma)+\lambda_k t)$ as a majorant, everything proceeds as before. In the case in which we are considering components $x^q{}_h(t)$ corresponding to λ_k, we obtain after integration by parts an expression of the form

$$\exp(\lambda_k t)\cdot\{\sum_{i=0}^{m} t^{m-i}a^{q+1}(t)/[(m-i)!(q\gamma)^{i+h+1}] + \sum_{i=0}^{k} t^i a^{q+1}(0)/[i!(q\gamma)^{m+h+i}]\},$$

$(k \leqq m)$. Therefore, we choose, as in the case of simple roots, a set of initial coordinates for these components so as to cancel out the set of terms due to the lower limit (i. e. 0) of integration. As before, the convergence of the sum of these initial coordinates follows from the convergence arguments for the series itself. Now, except for this special choice of coordinates at $t=0$ for certain components of $x^q(t)$, the argument proceeds as in the previous cases except that now the bound on our series is of the form

$$\exp(2nMn^2/(\gamma) + \lambda_k t) \sum_{i=0}^{m} t^{m-i}/(m-i)\,!.$$

This completes the proof of Theorem 2.

An immediate consequence of the above theorem and the work of Favard [6] is the following corollary.

COROLLARY. *If for the system $x' = A(t)x + B(t)$, where $B(t)$ is an* a. p. *vector and $A(t)$ satisfies the conditions of Theorem 2, the associated A_0 has roots all of whose real parts are nonzero, then this system has a unique* a. p. *solution.*

Part II. In this section, we consider the question of the existence of a. p. solutions for systems of the form $x' = A(t)x + B(t)$, where $A(t)$ is an $n \times n$ matrix and $B(t)$ an n vector of a. p. functions (Theorem 3). Before considering the statement of the theorem and its proof, it will be convenient to introduce several definitions for systems of the form $x' = [A + C(t)]x + B(t)$. Let $\lambda_j = \alpha_j + i\eta_j$, $j = 1, \cdots, h$, denote the characteristic roots of A, $m = \min |\alpha_j|$, and p be the number of λ_j that are conjugate roots of A. From now on, we assume that the parameters have been changed so that $m \geqq 1$, and we denote the multiplicity of λ_j by m_j. If c denotes $\underset{t,i,j}{\text{l. u. b.}} |c_{ij}(t)|$, then the system $x' = [A + C(t)]x + B(t)$ is said to satisfy the inequality I if for some $\epsilon > 0$, one has $m - \epsilon > c \sum_{j=1}^{h} (p + m_j)(m_j)^2$. The statement of Theorem 3 now takes the following form.

THEOREM 3. *Consider the system $x' = (A + C(t))x + B(t)$, where $B(t)$ and $C(t)$ have real* a. p. *entries. If this system satisfies inequality I, defined above, then it possesses a unique* a. p. *solution whose module is contained in the module of $[C(t), B(t), \mathrm{Im}(\lambda_j)]$.*

Proof. We first consider the case in which A has simple roots. We assume a solution of the form $x(t) = x^0(t) + \sum_{i=1}^{\infty} x^i(t)$, where the $x^i(t)$ are defined recursively as follows: $[x^0(t)]' = Ax^0(t) + B(t)$ and $[x^i(t)]' = Ax^i(t) + C(t)x^{i-1}(t)$ for $i > 0$. For convenience, we assume that $\mathrm{Re}(\lambda_i) > 0$ for $i > j$ and $\mathrm{Re}(\lambda_i) < 0$ for $i \leqq j$. In the formula for the variation of constants for the i-th components, we choose the limit of integration as ∞ $(-\infty)$ if $i > j$ $(i \leqq j)$. Thus, if we denote by k the $\underset{\text{all } l}{\text{l. u. b.}} |B_l(t)|$, we have for the i-th component of $x^0(t), x^0{}_i(t) = \int_t^{\infty} \exp(\lambda_h(t-s))B_h(s)\,ds$;

hence $|x^0{}_i(t)| \leqq k/m$. In the case of a conjugate root λ_l, we have for the k-th component $\int_t^\infty \exp[\lambda_l(t-s)][B_k(s)\cos\eta_k(t-s)+B_{k+1}(s)\sin\eta_l(t-s)]ds.$ Thus the componentwise bound now takes the form $|x^0{}_k(t)| \leqq 2k/m$. For the induction step, we assume

[a] $$\|x^{l-1}(t)\| \leqq [k(n+p)/m]\cdot[((n+p)c)/m]^{l-1};$$

then by the formula for variation of constants,

[b] $$|x^l{}_s(t)| \leqq [kc(n+p)/m][((n+p)c/m]^{l-1},$$

or

[c] $$|x^l{}_s(t)| \leqq [2kc(n+p)/m][((n+p)c)/m]^{l-1}$$

in the case of a conjugate root. Thus

[d] $$\|x^l(t)\| \leqq [k(n+p)/m][(n+p)c/m]^l,$$

and the induction is complete. By assumption, $(n+p)c/m = \delta < 1$, and the series for $x(t)$ is majorized by

$$[(n+p)k/m][1+\delta+\cdot\cdot\cdot] = (n+p)k/[m(1-\delta)]$$

and so converges uniformly. By Lemma 1, the $x^l(t)$ are a. p. and their modules are contained in the module described in the theorem. Since the uniform limit of a. p. functions is again a. p., the proof of the theorem is complete in the case of distinct roots. In the case of multiple roots, the procedure is exactly the same except that now one employs Lemma 1′ and uses the fact that $m > 1$, so that $|k|/m^l < |k|/m$ for any integer l. Then if $J = \sum_{j=1}^h (m_j+p)(m_j)^2/m$, the estimates [a], [b], [c], and [d] take the form

[a′] $$\|x^{l-1}(t)\| \leqq kJ(Jc)^{l-1},$$

[b′] $$|x^l{}_s(t)| \leqq kJ(Jc)^{l-1}c(m_j)^2,$$

[c′] $$|x^l{}_s(t)| \leqq 2kJ(Jc)^{l-1}c(m_j)^2,$$

[d′] $$\|x^l(t)\| \leqq kJ(Jc)^l,$$

and the proof proceeds as in the case of simple roots. This completes the proof of Theorem 3.

The above result introduces the following problem. Given a matrix $B(t)$, is it possible to find a constant matrix A such that, if δ denotes the l. u. b. $|a_{ij}-b_{ij}(t)|$, then the inequality I′) $m-\epsilon > \delta\sum_{j=1}^h (p+m_j)(m_j)^2$ holds

for some positive ϵ, where p and m_j are as in Theorem 3? The matrix that immediately suggests itself is $A^0 = (a^0_{ij})$, where

$$a^0_{ij} = [\text{l. u. b.}\, a_{ij}(t) + \text{g. l. b.}\, a_{ij}(t)]/2,$$

since this will minimize the δ which will be multiplied by at least n. Thus we have:

COROLLARY. *If the roots λ_j of the equation $x^n + a^0_{n-1}x^{n-1} + \cdots + a^0_0 = 0$ associated with the system $x^{(n)} + p_{n-1}(t)x^{(n-1)} + \cdots + p_0(t) = Q(t)$ have nonzero real parts and if* $\min |\operatorname{Re}\lambda_j|$ *satisfies the inequality* I′ *above, the system has a unique* a. p. *solution.*

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TAME COVERINGS AND FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES.*

Part I: Branch loci with normal crossings; Applications: Theorems of Zariski and Picard.

By Shreeram Abhyankar.

Introduction. In this paper, we shall study the fundamental group of an algebraic variety V minus a subvariety W over an arbitrary ground field, the classical case being subsumed as a special case. This will be done via first studying finite alegbraic coverings of V with branch loci contained in W. Here in the introduction, we shall only approximately describe the situation and indicate some of the results.

The finite galois groups over V of all tame (for definition see Section 2) finite galois coverings of V—isomorphic coverings being identified—with branch loci contained in W form an inverse system $\pi'(V-W)$ of a special kind which we shall call a group tower. A group G is said to be a weak parent group of a group tower π if G can be topologized so that the group tower of all continuous finite homomorphic images of G is isomorphic to π; if π is isomorphic to the group tower of *all* finite homomorphic images of G (i.e. if G is regarded as a discrete group), then G is said to be a parent group of π.[1] The possible existence of a finitely generated parent group (or somewhat weaker: the possible existence of a finitely generated weak parent group) of $\pi'(V-W)$ is the abstract analogue of the statement (Section 16) that in the classical case the topological fundamental group $\pi_1(V-W)$ is finitely generated; and hence if such a finitely generated parent (respectively, weak parent) group exists, we shall call it a tame fundamental parent (respectively, weak parent) group of $V-W$. Now one main result of this paper (Section 12) is that if V is nonsingular and simply connected, if W has only normal crossings and if the irreducible components of W move in linear systems of dimension greater than one, then denoting the number of these

* Received April 23, 1958.

[1] Here we chose to overlook a condition that the intersection of subgroups of G of finite index be 1, see Section 5; and for the corresponding situation in the classical case, see Section 17.

components by t, we have that $V - W$ has a tame fundamental weak parent group generated by t generators and that *any* tame fundamental weak parent group (and hence, in particular, any tame fundamental parent group) of $V - W$ is t-step nilpotent and is abelian in case the irreducible components of W are pairwise connected, i. e. any two have a point in common. As a corollary of this, we deduce the abstract version of a theorem of Zariski which in the classical case says that the fundamental group of a complex projective space minus a hyperplane with normal crossings and t irreducible components is an abelian group with t generators and one relation. As another corollary, we deduce the abstract version of a theorem of Picard which says that any 'cyclic' surface in the complex projective three space is simply connected.

Now in the classical case, the recent work of Grauert and Remmert shows that any finite unramified topological covering of $V - W$ can be completed to an alegbraic covering of V with branch locus contained in W, and hence $\pi_1(V - W)$ is a parent group of $\pi'(V - W)$. Therefore the above results tell us that in the classical case, $\pi_1(V - W)$ is respectively t-step nilpotent, and abelian.[1]

Our tools are mainly these: (1) Galois theory of local rings, including the concepts of splitting and inertia groups which are the higher dimensional analogues of the corresponding concepts in algebraic number theory and are due to Krull. These and other aspects of the galois theory of local rings were further developed by us in our previous work. In Section 2, we bring together, in suitable form, concepts and results to be used from this theory. (2) From algebraic geometry proper, we use mainly three things all due to Zariski, namely (i) normalization in an algebraic extension, (ii) generalized Bertini theorem, and (iii) degeneration principle. (3) Results from the local theory of normal crossings developed by us elsewhere and summarized in Section 3; this in part is the arithmetization of local fundamental groups; in our previous treatment of this, we had used Zariski's theorem on 'purity of branch locus,' while in our forthcoming new treatment which makes the theory valid also in the Kroneckerian case, we shall use Chow's recent work on 'connectedness' and 'local Bertini theorem.' (4) From topology, we derive our motivation for the fundamental group towe., etc. All these tools put together enable us to study fundamental groups in the abstract case when no classical topological techniques are available.

In forthcoming papers, we shall study fundamental groups of algebraic varieties when the branch loci have higher singularities, and there, in addition to the present tools, we shall employ analysis of singularities, quadratic trans-

formations and a concept of systems of curves with assigned singularities; and as an application, we shall obtain theorems on the nonexistence of irreducible plane curves of a given degree with prescribed singularities. Also in another paper, we shall develop a 'ramification theory' for complex manifolds and obtain similar results for fundamental groups of complex manifolds; in that set up, the role played in the algebrogeometric situation by the generalized Bertini theorem and the degeneration principle will be played by a result of Stein and a result of Serre.

For the sake of brevity, definiteness and clarity of exposition, in this paper, we shall deal with algebraically closed ground fields and projective varieties. That most of the relevant material of this paper goes over for nonalgebraically closed ground fields or for the complete abstract varieties of Weil will be shown in a later communication.

The contents of the various chapters and sections should be clear from their titles.

In concluding this introduction, I wish to express my great appreciation of Professor Oscar Zariski; for in the first place, the present investigations began in trying to carry over to the abstract case Zariski's classical theorem on fundamental groups mentioned above; and in the second place, as indicated, some of the important tools in this study are due to Zariski's foundations of abstract algebraic geometry.

TABLE OF CONTENTS

A. Algebrogeometric Preliminaries.

1. Conventions and notations. A ring R (always commutative with identity) will be said to be normal if it is integrally closed in its total ring of quotients. By "(R, M) is a local ring," we shall mean that R is a (not necessarily noetherian) local ring and M is its maximal ideal. Let (R, M) be a normal local ring with quotient field K, let K^* be a finite separable algebraic extension of K; recall that the integral closure of R in K^* has only a finite number of maximal ideals, the quotient rings with respect to these maximal ideals will be called the *local rings in* K^* *lying above* R; if (R^*, M^*) is a local ring in K^* lying above (R, M), then $R = K \cap R^*$ and $M = K \cap M^*$, and hence we may call (R, M) the local ring in K lying below (R^*, M^*); also recall that if K^*/K is galois, then the set of local rings in K^* lying above R is a complete set of K-conjugates. A field extension K^*/K will be said to be galois if it is normal algebraic and separable; $G(K^*/K)$ will then denote the galois group of K^*/K; unless otherwise stated, galois will mean finite galois. As usual, for a finite alegbraic extension K^*/K, $[K^*:K]_s$ and $[K^*:K]_i$ denote, respectively, the separable and the pure inseparable degrees of K^*/K. For a subgroup H of a group G, the index of H in G will be denoted by $[G:H]$; in particular, $[G:1]$ is the order of G.

Let V be an irreducible algebraic variety in a projective space over an algebraically closed ground field k and let $K = k(V)$, i.e. let K/k be a function field of V/k; then V is a projective model of K/k and is given by projectively related affine coordinate rings with quotient field K/k. Unless otherwise stated, we shall consider only those irreducible subvarieties (respectively, points) of V which are defined (respectively, rational) over k, thus the irreducible subvarieties of V will be in one to one correspondence with the quotient rings of the various affine coordinate rings of V in K/k. For an irreducible subvariety W of V, we shall denote the local ring of W on V by $Q(W, V)$, and the maximal ideal of $Q(W, V)$ by $M(W, V)$; note that k is a subfield of $Q(W, V)$ and K is the quotient field of $Q(W, V)$. V will be said to be normal if the local ring of each point (and hence, of each irreducible subvariety) of V is normal. Unless otherwise stated, terms like 'connected,' etc. will refer to the Zariski topology (over the ground field under consideration).

2. Galois theory of local rings and branch loci. In this section, we shall collect together, in suitable form, concepts and results on galois theory of local rings and branch loci. These will be based on Section 2 of [A2],[2]

[2] References in square brackets refer to the References at the end of the paper.

Section 1 of [A1], Sections 1, 5 and 6 of [A3], and when proofs are not given, they will be found in these references.

Let K^*/K be a galois extension, let (R, M) be a normal local ring with quotient field K and let (R^*, M^*) be a local ring in K^* lying above R. Recall that

$G_s(R^*/R) = G_s(R^*/K) =$ *splitting group* of R^* over R (or K) = group of automorphisms t of K^*/K for which $t(R^*) = R^*$.

$G_i(R^*/R) = G_i(R^*/K) =$ *inertia group* of R^* over R (or K) = group of automorphisms t of K^*/K for which $r \in R^*$ implies that $t(r) - r \in M^*$.

The fixed fields of $G_s(R^*/R)$ and $G_i(R^*/R)$ are respectively called the *splitting* and *inertia fields* of R^* over R (or of R^* over K).

LEMMA 1. *Let $G = G(K^*/K)$, $G_s = G_s(R^*/R)$, $G_i = G_i(R^*/R)$. Then*

(i) $[G : G_s] =$ *number of local rings in K^* lying above R.*

(ii) *G_i is a normal subgroup of G_s and* $[G_s : G_i] = [R^*/M^* : R/M]_s$.

(iii) $[G_i : 1] = [K^* : K]$ [number of local rings in K^* lying above $R]^{-1}[R^*/M^* : R/M]_s^{-1}$.

LEMMA 2. *Let K' be a subfield of K^* which is galois over K, and let (R', M') be a local ring in K' lying above (R, M) such that R^* lies above R'. Then* (i) *the natural homomorphism of $G(K^*/K)$ onto $G(K'/K)$ induces a homomorphism of $G_s(R^*/R)$ onto $G_s(R'/R)$ with kernel $G_s(R^*/R')$ and a homomorphism of $G_i(R^*/R)$ onto $G_i(R'/R)$ with kernel $G_i(R^*/R')$. Hence in particular:*

(ii) $$[G_s(R^*/R) : 1] = [G_s(R^*/R') : 1][G_s(R'/R) : 1] \text{ and}$$
$$[G_i(R^*/R) : 1] = [G_i(R^*/R') : 1][G_i(R'/R) : 1].$$

Proof. (i) follows from a straightforward application of galois theory in view of the equations: $R' = R^* \cap K'$ and $M' = M^* \cap K'$. (ii) follows from (i), and can also be proved directly thus: Since $[K^* : K] = [K^* : K'][K' : K]$; $[R^*/M^* : R/M]_s = [R^*/M^* : R'/M']_s[R'/M' : R/M]_s$; and the local rings in K^* lying above R are exactly the local rings in K^* lying above the various local rings in K' which lie above R; in view of Lemma 1, it is enough to show that if $\bar{R}^*$ is any local ring in K^* lying above R and $\bar{R}' = \bar{R}^* \cap K'$, then $G_s(\bar{R}^*/\bar{R}')$ and $G_i(\bar{R}^*/\bar{R}')$ are of the same orders, respectively, as $G_s(R^*/R')$ and $G_i(R^*/R')$. Since R^* and $\bar{R}^*$ are K-conjugate, there exists $t \in G(K^*/K)$

with $t(R^*) = \bar{R}^*$. Let u be an element of $G(K^*/K')$ considered as a subgroup of $G(K^*/K)$. Then $g \in G_s(\bar{R}^*/\bar{R}')$ if and only if $g(\bar{R}) = \bar{R}^*$, i.e., $g(t(R^*)) = t(R^*)$, i.e., $(t^{-1}gt)(R^*) = R^*$, i.e., if and only if $t^{-1}gt \in G_s(R^*/R')$. Hence $G_s(\bar{R}^*/R')$ and $G_s(R^*/R')$ are conjugate in G, and similarly for the inertia groups.

LEMMA 3. *Let $K_1, K_2, \cdots, K_t$ be subfields of K^* such that K^* is their compositum and each of them is galois over K. Let R_j be a local ring in K_j lying above R. Then $G_i(R^*/R)$ is isomorphic to a subgroup of the direct sum of $G_i(R_1/R), G_i(R_2/R), \cdots, G_i(R_t/R)$ whose natural projection on each of the components $G_i(R_j/R)$ is onto; hence, in particular, for each j, the order of $G_i(R_j/R)$ divides the order of $G_i(R^*/R)$ and the latter divides the product of the orders of $G_i(R_1/R), \cdots, G_i(R_t/R)$. Similar statements hold for the splitting groups.*

Proof. Apply inductions on t and use Lemma 2 repeatedly.

Now let K be an algebraic function field over an algebraically closed field k of characteristic p, let K^* be a finite separable algebraic extension of K, let V be a normal projective model of K/k, let V^* be a K^*-normalization of V and let ϕ be the natural map of V^* onto V. Recall that V^* is characterized, up to natural biregular maps, by the property that for any irreducible subvariety W of V, if $W^*_1, \cdots, W^*_t$ denote the irreducible components of $\phi^{-1}(W)$, then $Q(W^*_1, V^*), \cdots, Q(W^*_t, V^*)$ are exactly the local rings in K^* lying above $Q(W, V)$. Concepts and adjectives defined for $Q(W^*_j, V)$ over $Q(W, V)$ will be applied to W^*_j over V (or over K); thus, for instance, if K^*/K is galois, then the inertia group of $Q(W^*_j, V)$ over $Q(W, V)$ will be called the inertia group of W^*_j over V or over K and will be denoted by $G_i(W^*_j/V)$ or $G_i(W^*_j/K)$. Strictly speaking, these concepts depend on the particular rational map ϕ of V^* onto V, i.e. on the particular embedding of $k(V)$ into $k(V^*)$; however, this will be always clear from the context.[3]

Let W^* be an irreducible subvariety of V^*, let $W = \phi(W^*)$, let (R^*, M^*) and (R, M) be the completions, respectively, of $Q(W^*, V^*)$ and $Q(W, V)$ canonically assuming that R is a subring of R^*, let E^* and E be the quotient

[3] For visual and mental facility, by an abuse of language, when the reference to V is clear from the context and there is no cause for confusion, we shall allow ourself to apply concepts and adjectives for W^*_j over V as if they were for W^*_j over W; thus, for instance, we may write $G_i(W^*_j/W)$ for $G_i(W^*_j/V)$, or we may call $r(W^*: V)$ the ramification index of W^* over W and denote it by $r(W^*: W)$. [Note that if U^* is an irreducible subvariety of V^* containing W^* and $U = \phi(U^*)$, then $r(W^*/W)$ will in general depend on whether this is over U or V].

fields of R^* and R respectively, let E' be a least galois extension of E containing E^*, and let R' be the integral closure of R in E'. Since R is complete, R' is a local ring and the only one in E' lying above R; let M' be the maximal ideal in R'. We set

$$r(W^*:V) = \textit{ramification index of } W^* \textit{ over } V \textit{ (or over } K\textit{)}$$
$$= [E^*:E][R^*/M^*:R/M]_s^{-1}.^3$$

We shall say that W^* is *ramified for* ϕ (or over V, or over K) if $r(W^*:V) \neq 1$, i.e., $r(W^*:V) > 1$, and we shall say that W^* is tamely ramified for ϕ (or over V, or over K) if

$$[E':E][R'/M':R/M]_s^{-1} \not\equiv 0 \pmod p$$

in case $p \neq 0$ (and no restriction if $p = 0$).[4] W will be said to be *ramified* for ϕ^{-1} (or for V^*, or in K^* since this is independent of the model V^*) if some irreducible component of $\phi^{-1}(W)$ is ramified for ϕ. Again, W will be said to be *tamely ramified* for ϕ^{-1} (or for V^*, or in K^*) if each irreducible component of $\phi^{-1}(W)$ is tamely ramified for ϕ. If each irreducible subvariety of V is tamely ramified in K^*, then K^* will be said to be a *tamely ramified extension of* V, or *tamely ramified over* V, or V^* will be said to be *a tamely ramified covering of* V. The set of points of V which are ramified in K^* will be called *the branch locus of* V *in* K^* or the branch locus of K^* (or V^*) over V, or the branch locus of ϕ^{-1} and *will be denoted by* $\Delta(K^*/V)$ or $\Delta(V^*/V)$. Since k is algebraically closed, $\Delta(K^*/V)$ is the set of points P of V for which $\phi^{-1}(P)$ consists of less than $[K^*:K]$ points. If $\Delta(K^*/V)$ is empty, then we shall say that K^* *is unramified over* V or that V^* *is an unramified covering of* V.

LEMMA 4. *$\Delta(K^*/V)$ is a proper subvariety of V, and an irreducible subvariety of V is ramified in K^* if and only if it is contained in $\Delta(K^*/V)$.*

Now let K' be a least galois extension of K containing K and let W' be an irreducible subvariety corresponding to W on a K'-normalization V' of V.

LEMMA 5. $\Delta(K'/V) = \Delta(K^*/V)$.

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[4] In [A3], we have defined 'tamely ramified' to mean $r(W^*:V) \not\equiv 0 \pmod p$. As was shown in Section 5 of [A3], these two definitions coincide for curves (over an algebraically closed ground field). However, for higher dimensional varieties, the definition given in [A3] is not the correct one and the present definition should be substituted in [A3]; for otherwise, Remark 9 of Section 9 of [A3], which is the generalization of Theorem 5 there to higher dimensional varieties, need not be true. Also note that 'tamely ramified' includes 'unramifield.'

Invoking Lemma 5 of Section 2 of [A2], Lemma 1 of Section 5 of [A3], and Lemmas 1, 3, and 5 of this section, we get the following two lemmas.

LEMMA 6. *$r(W':W)=[G_i(W'/W):1]$; hence W is ramified in K' if and only if $[G_i(W'/W):1]\neq 1$, and W is tamely ramified in K' if and only if $[G_i(W'/W):1]\not\equiv 0 \pmod p$ in case $p\neq 0$ (and trivially always for $p=0$).*

LEMMA 7. *W is tamely ramified in K' if and only if W is tamely ramified in K^*.*

Hence 'ramified' and 'tamely ramified' for an irreducible subvariety of V can be defined without passing to completions of the local rings. Also for galois extensions K'/K, 'ramification index,' 'ramified,' and 'tamely ramified' can be defined for irreducible subvarieties of V as well as those of V' without passing to completions of the local rings.

LEMMA 8. *Let $K\subset K_1\subset K_2$ be finite separable algebraic extensions and let V_1 be the K_1-normalizations of V. If K_1/V is tamely ramified and K_2/V_1 is tamely ramified, then K_2/V_1 is tamely ramified.*

Proof. Use Lemma 4 of Section 2 of [A2].

LEMMA 9. *Let K^* and K_1 be finite separable algebraic extensions of K and let K^*_1 be a compositum of K^* and K_1. Let V^* be a K^*-normalization of V. If K_1/V is unramified (respectively, tamely ramified), then K^*_1/V^* is unramified (respectively, tamely ramified).*

Proof. In view of Lemmas 5 and 7, we may assume K^*/K, K_1/K and K^*_1/K are galois. Let W^*_1 be an irreducible subvariety on a K^*_1-normalization of V and let W, W^* and W_1 be the corresponding irreducible subvarieties of V, V^*, and a K_1-normalization of V. Then by Lemmas 3 and 6, we have that $r(W^*_1:W)$ divides $r(W^*:W)r(W_1:W)$, and by Lemma 4 of Section 2 of [A2], we have that $r(W^*_1:W)=r(W^*_1:W^*)r(W^*:W)$. Therefore $r(W^*_1:W^*)$ divides $r(W_1:W)$.

Part of Lemma 6 of Section 2 of [A2] is this:

LEMMA 10. *Let $U'\subset W'$ be irreducible subvarieties of V'. Then $G_s(W'/V)\subset G_s(U'/V)$ and $G_i(W'/V)\subset G_i(U'/V)$.*

Lemmas 7 and 10 give the following:

LEMMA 11. *K^*/V is tamely ramified if and only if each point of V is tamely ramified in K^*.*

Lemmas 2, 3 and 7 give the following lemma.

LEMMA 12. *Let $K_1 \subset K_2$ and also $L_1, L_2, \cdots, L_h$ be finite separable algebraic extensions of K and let L be a compositum of $L_1, L_2, \cdots, L_h$. Let D be a subvariety of V. Then*

(1) *$\Delta(K_2/V) \subset D$ implies $\Delta(K_1/V) \subset D$; and $\Delta(L_j/V) \subset D$ for $j = 1, \cdots, h$ implies $\Delta(L/V) \subset D$. Also,* (2) *K_2 is tamely ramified over V implies K_1 is tamely ramified over V; and L_j is tamely ramified over V for $j = 1, \cdots, h$ implies L is tamely ramified over V.*

Lemmas 4 and 5 of Section 2 of [A2] imply the following:

LEMMA 13. *Let K_1 be a field contained between K and the inertia field of W' over W, and let W_1 be the irreducible subvariety corresponding to W' on a K_1-normalization of V. Then W_1 is unramified over V.*

Since a one dimensional normal local domain is a discrete valuation ring, we have

LEMMA 14. *If W' is an irreducible subvariety of V' of codimension* 1 *and if K' is tamely ramified over V, then $G_i(W'/V)$ is cyclic and its order is prime to p in case $p \neq 0$.*

Let v be a real valuation of K/k and let v' be a K'-extension of v. By $G_s(v'/v) = G_s(v'/K)$ and by $G_i(v'/v) = G_i(v'/K)$, we shall denote, respectively, the *splitting* and *inertia groups* over K of the valuation ring of v'. We shall say that v is *ramified* in K' if $G_i(v'/v) \neq 1$, also, we shall say that v is tamely ramified in K' if $[G_i(v'/v):1] \not\equiv 0 \pmod{p}$ in case $p \neq 0$, (since K'/K is galois, this does not depend on v'). Then a part of Lemma 6 of Section 2 of [A2] gives

LEMMA 15. *If K' is unramified over V, then every real valuation of K/k is unramified in K'. If K' is tamely ramified over V, then every real valuation of K/k is tamely ramified in K'.*

Zariski's theorem on 'purity of branch locus' says the following:

LEMMA 16. *If P is a simple point of V, then at P, $\Delta(K^*/V)$ is of codimension* 1 (*and hence of pure codimension* 1).

The proof of this given in [A1, Theorem 1] is incorrect. Professor Zariski has a correct proof (to be published), and we have a proof of the following weaker form which will be published in [A5].

LEMMA 17. *If P is a simple point of V and if P is tamely ramified in K^*, then at P, $\Delta(K^*/U)$ is of codimension 1 (and hence of pure codimension 1).*

Lemma 16 will not be used in this paper except for an incidental observation in Section 10 (Lemma 30).

3. Local theory of normal crossings. Let (R, M) be the local ring of a simple point P on an irreducible n dimensional algebraic variety V, let W be a subvariety of V which is pure $n-1$ dimensional at P and let $W_1, W_2, \cdots, W_t$ be the irreducible components of W passing through P. Let q_j and q be the ideals at P, respectively, of W_j and W; i.e., $q_j = R \cap M(W_j, V)$ and $q = q_1 \cap q_2 \cap \cdots \cap q_t$. Let $(\bar{R}, \bar{M})$ be a completion of (R, M). We shall say that W has an *h-fold normal crossing at P* if there exists a minimal basis $x_1, x_2, \cdots, x_n$ of $\bar{M}$ such that

$$q_1\bar{R} = x_1x_2\cdots x_{u_1}\bar{R}, \qquad q_2\bar{R} = x_{u_1+1}x_{u_1+2}\cdots x_{u_2}\bar{R}, \cdots,$$
$$q_t\bar{R} = x_{u_{t-1}+1}x_{u_{t-1}+2}\cdots x_{u_t}\bar{R}; \text{ with } 0 < u_1 < u_2 < \cdots < u_t = h. \qquad \cdots\cdots\cdots (A).$$

It is clear that the index h is uniquely determined by W at P, i.e., it is independent of the basis $x_1, \cdots, x_n$. Note that R is a unique factorization domain, and hence each q_j is a principal ideal and q is also equal to $q_1q_2\cdots q_t$. We assert that W has an h-fold normal crossing at P if and only if any one of the following conditions holds: (1) There exists a minimal basis $x_1, \cdots, x_n$ of $\bar{M}$ with $x_{h+1}, x_{h+2}, \cdots, x_n$ in R such that we have (A) with $\bar{R}$ replaced by R. (2) There exists a basis $x_1, \cdots, x_n$ of $\bar{M}$ with $x_{h+1}, x_{h+2}, \cdots, x_n$ in R such that $q = x_1x_2\cdots x_hR$. (3) There exist elements $x_{h+1}, x_{h+2}, \cdots, x_n$ in R such that $qx_{h+1}x_{h+2}\cdots x_n\bar{R}$ is generated by the product of the elements in some minimal basis of $\bar{M}$. For assume that there exists a minimal basis $x_1, \cdots, x_n$ of $\bar{M}$ such that (A) holds. Fix v_j in R with $v_jR = q_j$ for $j = 1, \cdots, t$. Then the generator of $q_j\bar{R}$ exhibited in (A) differs from v_j by a multiplicative unit in $\bar{R}$, and hence by multiplying $x_1, \cdots, x_h$ by suitable units in $\bar{R}$, we may assume that the generators of $q_j\bar{R}$ exhibited in (A) coincides with v_j for $j = 1, 2, \cdots, t$. Now there exists y_j in R with $y_j \equiv x_j \pmod{\bar{M}}$. Then $x_1, x_2, \cdots, x_h, y_{h+1}, y_{h+2}, \cdots, y_n$ is also a minimal basis of $\bar{M}$, and hence we could assume that x_j is in R for $j = h+1, h+2, \cdots, n$; this gives (1), and (2) and (3) at once follow from (1). Now we shall prove the converse. (1) trivially implies that W has an h-fold normal crossing at P. That (2) and (3) imply that W has an h-fold normal crossing at P follows from the

fact that $\bar{R}$ is a unique factorization domain and that the elements of any minimal basis of $\bar{M}$ are mutually prime irreducible nonunits of $\bar{R}$.

W will be said to have *a strong h-fold normal crossing at P* if in (A), we have $u_1 = u_2 - u_1 = u_3 - u_2 = \cdots = u_t - u_{t-1} = 1$ and hence $t = h$. In view of conditions (1) and (2) above, this is equivalent to saying that there exists a minimal basis $x_1, x_2, \cdots, x_n$ of M such that

$$(1^*) \quad q_j = x_j R \text{ for } j = 1, 2, \cdots, t = h; \qquad \text{or } (2^*) \quad q = x_1 x_2 \cdots x_h R.$$

Geometrically speaking, W has a normal crossing at P if P is a simple point of each analytic sheet of W and the tangent hyperplanes to these sheets at P are linearly independent; if the number of sheet is h, then the normal crossing is h-fold. Furthermore, the normal crossing is strong if P is a simple point for each algebraic component of W so that at P, the number of algebraic components of W is equal to the number of analytic sheets of W. Also note that what we called a normal crossing in [A1] is now being called a strong normal crossing.

Now let K be an n dimensional algebraic function field over an algebraically closed ground field k of characteristic p, let K^* be a galois extension of K, let V be a normal projective model of K/k, let V^* be a K^*-normalization of V, let ϕ be the rational map of V^* onto V, let P^* be a point of V^*, let $P = \phi(P^*)$; assume that P is a simple point of V, that P is tamely ramified in K^* and that $\Delta(K^*/V)$ has a normal crossing at P. Observe that since k is algebraically closed, we have that $G_i(P^*/P) = G_s(P^*/P)$. Now we assert the following:

PROPOSITION 1. *$G_i(P^*/P)$ is abelian.*

From Proposition 1, one can deduce the following:

PROPOSITION 2. *Let W be an irreducible component of $\Delta(K^*/V)$ through P. Then W does not split (locally) at P^*,* i. e., *only one irreducible component of $\phi^{-1}(W)$ passes through P^*.*

Now Proposition 1, in case of strongly normal crossings, was proved in Theorem 2 of [A], the proof there was based on 'purity of branch locus,' i. e., Lemma 16 or Theorem 1 of [A1], or in fact, only on 'purity of branch locus for tame coverings,' i. e., Lemma 17. The proof of Theorem 1 of [A1] is incorrect. For not necessarily strong normal crossings, one could adapt the proof of [A1], provided one has 'purity' also for algebroid varieties. Professor Zariski's new proof (unpublished) of 'purity' is believed to be also applicable for the algebroid case. However, in a forthcoming paper [A5],

we shall prove Proposition 1 and 2 directly (and simultaneously), deriving 'purity' (for tame coverings) as an incidental corollary; the treatment there will be applicable to algebraic, algebroid, as well as Kroneckerian varieties; in that treatment, we shall use Chow's recent work on 'connectedness' and 'local Bertini theorem' [C2, 3]. Here we shall briefly indicate the idea of how, for strong normal crossings, one can deduce Proposition 2 from Proposition 1. Let then $W = W_1, W_2, \cdots, W_h$ be the irreducible components of $\Delta(K^*/V)$ at P; let (R, M) be the local ring of P on V; choose a minimal basis $x_1, \cdots, x_n$ of M such that $x_j R$ is the ideal of W_j at P. If the assertion is proved for an algebraic extension of K^*, then it will *a fortiori* imply it for K^*, and hence we may replace K^* by a suitable algebraic extension. Making considerations as in Section 2 of [A1], Proposition 1 will imply that (extending K^* suitably) we may arrange matters so that the completions of $Q(P^*, V^*)$ and $Q(P, V)$ are, respectively,

$$\bar{R}^* = k[[x_1^{1/m}, x_2^{1/m}, \cdots, x_h^{1/m}, x_{h+1}, x_{h+2}, \cdots, x_n]]$$

and $\bar{R} = k[[x_1, x_2, \cdots, x_n]]$, where $m \not\equiv 0 \pmod{p}$ in case $p \neq 0$.[5] The matter being 'local,' we may replace K^*/K by E^*/E, where E^* and E are the quotient fields of $\bar{R}^*$ and $\bar{R}$ respectively. Now it is enough to observe that $x_1\bar{R}^*$ is prime, or rather that the valuation v_1 given by $x_1\bar{R}$ does not split in E^*; for instance, because in E^*, v_1 is ramified to index m and acquires a separable residue field extension of degree $(h-1)m$.

We remark that Proposition 1, in the classical case, follows from the fact that the local topological fundamental group at a normal crossing is abelian.

4. Linear systems. In this section, we recall needed information on linear systems [Z4, 6]. Let K be an n dimensional algebraic function field over an algebraically closed ground field k and let V be a normal projective model of K/k. We shall use the notations and definitions of Section 2 of [Z6] with the following modification: (1) an $(n-1)$ cycle on V will be called a divisor; (2) we shall not operate in a universal domain, but shall operate within K/k, i.e., all the divisors (unless otherwise stated) will be rational over k, all the functions will be in K and so on. Let us, for instance, recall that for a positive divisor D on V, $|D|$ *denotes the complete linear system determined by* D, it is the linear system of all nonnegative divisors on V which are linearly equivalent to D, and we have

[5] Actually, in [A1], the procedure was reversed, i.e., first it was shown that $\bar{R}^*$ and $\bar{R}$ can be arranged thus, and from it, the abelian character of $G_i(P^*/P)$ was deduced.

$$\dim |D| = [k\text{-dimension of the vector space of all functions } f \text{ in } K \text{ for which } (f) + D \geqq 0] - 1.$$

Part of the considerations of [Z4], especially Sections 14 and 15, can be stated in the following form:

"GENERALIZED THEOREM OF BERTINI." *Let L be a linear system free from fixed components and of dimension greater than 1. If L is composite with a pencil, i.e., if the rational map given by L maps V onto a curve, then each member of L is irreducible, and conversely, if L is not composite with a pencil, i.e., if the rational map given by L maps V onto a variety of dimension greater than 1, then L contains an irreducible member, or equivalently, a "generic" member of L is absolutely irreducible (here we are going outside our ground field k).*

Note that if D is a prime divisor with $\dim |D| \geqq 1$, *then $|D|$ is without fixed components.* Now let K^* be a finite separable algebraic extension of K, let V^* be a K^*-normalization of V, and let ϕ be the rational map of V^* onto V. Let L be a linear system on V without fixed components and of dimension greater than 1. Given D in L, write $D = m_1D_1 + \cdots + m_tD_t$, where the D_j are prime divisors and $m_j > 0$; let $D_{j1}, D_{j2}, \cdots, D_{jq_j}$ be the irreducible components of $\phi^{-1}(D_j)$; set $D^* = \sum_{j=1}^{t} m_j \sum_{h=1}^{q_j} r(D_{jh} : D_j) D_{jh}$; and let L^* be the set of divisors D^* on V^* spanned out as D ranges over L. Then it is easily verified that L^* is a linear system on V^* without fixed components and that the rational map of V^* given by L^* is the compositum of ϕ and the rational map of V given by L; hence L^* is composite with a pencil if and only if L is composite with a pencil. *We shall denote the linear system L^* by $\phi^{-1}(L)$ and shall call it the ϕ^{-1} image of L.*

B. Preliminaries on Group Towers.

5. Definitions. Let us recall the definition of an inverse system of groups [ES, Chapter VIII]: A *partially ordered set* S is a set with a relation $s < t$ which is transitive and reflexive such that $s < t$ and $t < s$ in S implies that $s = t$. A *directed set* S is a partially ordered set such that s, t in S implies that there exists u in S with $s < u$ and $t < u$. Let S' be a (partially ordered) subset of a directed set S; S' will be said to be a *cofinal subset* of S if s in S implies that there exists s' in S' with $s < s'$; S' will be said to be a *saturated subset* of S if $s < s'$ in S with s' in S' implies that s is in S'; note

that if S' is a saturated subset of S and S'' is a subset of S', then S'' is a saturated subset of S' if and only if S'' is a saturated subset of S, also if S' is a saturated and cofinal subset of S then S' must be S itself. An inverse system π *of groups* is a set $\{G_s\}$ of groups indexed by s running over a directed set S, together with homomorphisms $\alpha_s{}^t \colon G_t \to G_s$ for each $s < t$ in S, such that $\alpha_s{}^s =$ identity for s in S and $\alpha_s{}^t \alpha_t{}^u = \alpha_s{}^u$ for $s < t < u$ in S. If S' is a directed subset of S, the corresponding groups $\{G_s\}$ with s in S' form a *subsystem* π' of π; π' will be said to be a *cofinal* (*respectively: saturated*) *subsystem* of π if S' is a cofinal (respectively: saturated) subset of S'. Sometimes we shall take the set $\{G_s\}$ of groups as it own indexing set.

Let G be a group; let S be the set of all normal subgroups $s, t, \cdots$ of G with the relation $s < t$ if and only if $s \supset t$, now the intersection of any two normal subgroups is again a normal subgroup, and hence S *is a directed set*; when we talk of a partially ordered (in particular, directed) set S' of normal subgroups of a group G, we shall always be referring to this order relation; S' will be said to be a *aturated* (*respectively: directed, cofinal*) *set of normal subgroups of* G if S' is a saturated (respectively: directed, cofinal) subset of S. Let G be a group and let S' be a directed set of normal subgroups of G; by G/S', we shall denote the family of factor groups $\{G/s'\}$ of G indexed by s' running over S' and together with the natural onto homomorphisms; G/S' is then clearly an inverse system of groups; let S be the set of all normal subgroups of G, we shall call G/S *the derived inverse ysstem of* G; now G/S' is a subsystem of G/S, and it is a saturated (respectively: cofinal) subsystem if and only if S' is a saturated (respectively: cofinal) subset of S.

Let π be an inverse system of finite groups. Then π will be called a *group tower* if for each G_t in π, the inverse subsystem of π consisting of all G_s with $s < t$ is isomorphic to the derived inverse system of G_t under an isomorphism which is the identity on G_t. Observe that a subsystem of a group tower is a subtower if and only if it is a caturated subsystem; also note that a group tower has no cofinal subtowers other than itself. Now let G be a group, then the subset of all finite groups of the derived inverse system of G is a saturated subsystem [6] and it will be called the derived group tower of G; note that if S' is a set of normal subgroups of G, then G/S' is a group tower if and only if S' is a saturated set of normal subgroups of finite indices. If π is a group tower and p is a prime number, then the group in π

[6] (i) Any subgroup containing a subgroup of finite index is again of finite index. (ii) From the isomorphism theorem, it follows that the intersection u of any two normal subgroups s and t of finite indices is again of finite index; in fact, the index of u divides the product of the indices of s and t.

whose orders are prime to p form a subtower of π.[6] If G is a group, then the subtower of the derived group tower of all groups whose orders are prime to a given prime number p will be called *the modulo p derived group tower of G.*

Now let G be a group and let $\pi = \{G_{s^*}, \alpha_{s^*}{}^{t^*}, s^* \in S^*\}$ be a group tower. Suppose there exists a saturated set S of normal subgroups of G, a one to one order preserving map f of S onto the indexing set S^* of π [or onto the set $\{G_{s^*}\}$], and a family $\{f_s\}$ again denoted by f of onto homomorphisms $f_s\colon G \to G_{f(s)}$ [or respectively, $f_s\colon G \to f(s)$] such that if $s < t$ in S, then we have

$$\alpha_{f(s)}{}^{f(t)} f_t = f_s;$$

observe that then π is isomorphic to G/S. If the intersection of all the groups in S is 1 (i.e., contains only the identity element of G), then we shall say that f is *a weak parent map of G onto π, and S will be called the kernel of f.* Secondly, if S is the set of *all* normal subgroups of G of finite index and the intersection[7] of all the groups in S is 1, then we shall say that f *is a parent map of G onto π* and S will be called the kernel of f. Furthermore, for a prime number p, if S contains all the normal subgroups of G of finite index prime to p (and may contain some others) and if the intersection of all the normal subgroups of G of finite index is 1, then we shall say that f is *a modulo p quasi parent map of G onto π*, and S will be called the kernel of f. Finally, for a prime number p, if: (1) f is a modulo p quasi parent map of G onto π and (2) f is a weak parent map of G onto π, then we shall say that f is *a modulo p parent map of G onto π*, and S will be called the kernel of f; note that in the presence of (1), (2) is equivalent to: (2′) the intersection of all the groups in the kernel of f is 1.

Again let G be a group and let π be a group tower. Then G will respectively be said to be: (A) *a weak parent group of π*, (B) *a parent group of π*, (C) *a modulo p quasi parent group of π*, (D) *a modulo p parent group of π*; if there respectively exists: (A*) a weak parent map of G onto π, (B*) a parent map of G onto π, (C*) a modulo p quasi parent map of G onto π, (D*) a modulo p parent map of G onto π.

Now let G be a group, let N be the intersection of all normal subgroups

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[7] We may note the well known fact that for any group G, the intersection of all normal subgroups of finite index and the intersection of all subgroups of finite index coincide. For if K is a normal subgroup of G of finite index, then the intersection H of all conjugates of K is a normal subgroup and has finite index; for instance, see page 84 of [K1], or observe that H is the kernel of the natural homomorphism of G into the permutation group of left (or right) K-cosets.

of G of finite index, let π be a group tower, and observe the following: (A′) there exists a normal subgroup M of G such that G/M is a weak parent group of π if and only if there exists a set S of normal subgroups of G such that G/S is isomorphic to π. [*'If'*: Set $M=$ the intersection of all groups in S. *'Only if'*: G/M is a weak parent group of π implies that there exists a set S^* of normal subgroups of G/M such that $(G/M)/S^*$ is isomorphic to π and the intersection of all the groups in S^* is 1; let S be the set of all $\phi^{-1}(s^*)$ with s^* in S^*, where ϕ is the canonical homomorphism of G onto G/M.] $(B')\,G/N$ is a parent group of π if and only if π is isomorphic to the derived group tower of G. (C′) G/N is a modulo p quasi parent group of π if and only if there exists a subtower π' of the derived group tower of G such that π' contains the modulo p derived group tower of G, and π' is isomorphic to π. (D′) G/N is a modulo p parent group of π if and only if there exists a subtower π' of the derived group tower of G, such that π' contains the modulo p derived group tower of G, the intersection of "the kernels in G of the various groups in π'" is 1, and π' is isomorphic to π.

6. Topological considerations for group towers. In this section, we wish to make some incidental observations which easily follow from well known general facts [P, Chapter III; ES, Chapter VIII]. All topological groups considered will be assumed to be Hausdorff.

Remark 1. Let G be a group which is the weak parent group of some group tower, i. e., there exists a saturated set S of normal subgroups of G of finite index such that the intersection of all the groups in S is 1. Then G can be topologized by considering the members of S as a system of neighborhoods of the identity, and then S will coincide with the set of *closed* normal subgroups of G of finite index so that G/S will exactly be the group tower of all *continuous* finite homomorphic images of G. Conversely, if G is a topological group such that the intersection of all closed normal subgroups of G of finite index is 1, then G/S is a group tower, where S is the set of all closed normal subgroups of G of finite index. Thus we could give the following equivalent definition of weak parent groups (and call it a *topological parent group*): *A weak parent group of a group tower π is a topological group G in which the intersection of all closed normal subgroups of finite index is* 1 (*or equivalently the intersection of all closed subgroups of finite index is* 1) *such that π is isomorphic to the group tower of all continuous finite homomorphic images of G.* Note that such a group G must be totally disconnected, and also observe that for a topological group G, a sub-

group of finite index is closed if and only if it is open. *If G is to be a parent group, then the topology must be the one—or any other stronger than the one —obtained by taking for a system of neighborhoods of the identity all subgroups of finite index.*

Remark 2. Let $\pi = \{G_s, \alpha_s^t, s \in S\}$ be a group tower and let G_∞ be the inverse limit of π. Recall that G_∞ is the subgroup of $\prod_{s \in S} G_s$ consisting of elements $x = \{x_s\}$ for which $\alpha_s^t(x_t) = x_s$ for all $s < t$ in S. If we consider the G_s to be discrete topological groups, G_∞ becomes a compact group. G_∞ is a weak parent group of π since in G_∞, the intersection of closed normal subgroups of finite index is 1 and π is canonically isomorphic to the derived tower of continuous finite homomorphic images of G_∞. Let G' be any weak parent group of π with a weak parent map f of G' onto π and kernel S'. For g in G', set $\phi(g) = \{x_{f(s')}\}$, where $x_{f(s')} = f_{s'}(g)$. Then ϕ is easily seen to be an isomorphism of G' into G_∞. If we topologize G' by considering members of S as a system of neighborhoods of the identity, then ϕ is continuous and $\phi(G')$ is dense in G_∞. Conversely, a subgroup of G_∞ is dense if and only if the restriction of the natural projections of G_∞ into G_s maps G' onto G_s for all s in S, i.e., if and only if these projections give an isomorphism between π and the group tower of all continuous finite homomorphic images of G' thus making G' a weak parent group of π. Hence (up to a topological isomorphism) G_∞ is the only *compact "topological" parent group* of π.

Remark 3. Now assume that π is a group tower indexed by a countable set. Then the topologized inverse limit of π is compact totally disconnected and satisfies the second axiom of countability. Conversely, a compact totally disconnected topological group satisfying the second axiom of countability is canonically topologically isomorphic to the topologized inverse limit of its topological group tower (i.e., the group tower of continuous finite homomorphic images).

7. Existence of weak parent groups, parent groups, etc.

Example 1. Eilenberg and Zippin have communicated the following example which shows that in a topological compact totally disconnected group satisfying the second axiom of countability, a subgroup of finite index need not be closed; and in view of Remark 3 of Section 6, this shows that *the inverse limit of a group tower need not be its parent group under the natural map.* Let G be the product of countably infinite copies of a group Z_2 with

two elements 0 and 1. Considering Z_2 to be discrete, G then becomes compact totally disconnected and satisfies the second axiom of countability. Let G' be the corresponding direct sum considered as a subgroup of G. Then G' is dense in G. There exist subgroups H of G of index two containing G' and hence not closed in G. For instance, (i) consider Z_2 as a field and treat G as a vector space over Z_2, then any subgroup of G is a vector subspace and since $\dim G/G'$ is not zero (in fact is infinite), there are lots of subspaces H of G containing G' with $\dim(G/H) = 1$; or (ii) fix x in G not in G' and let H be a maximal element in the set of subgroups of G containing G' but not x.

Example 2. Let H be an infinite cyclic group, and let π be the modulo p derived group tower of H for a given prime number p; then it is clear that *H is a finitely generated weak parent group of π and hence a finitely generated modulo p parent group of π.* Suppose, if possible, that π has a finitely generated parent group G. Then (Proposition 4 of Section 9) G must be abelian. Since π contains a group of any order prime to p, it is clear that G cannot be finite. Hence G can be mapped homomorphically onto an infinite cyclic group and hence onto a finite cyclic group of any order including p. Therefore G cannot be a parent group of π. *Thus π has no finitely generated parent group.*

Remark 4. Note that the inverse limit G of a group tower π cannot be finitely generated unless G is finite, i. e., unless π contains only a finite number of groups. For since G is a compact topological group, G has a non-trivial left invariant Haar measure such that the measure of G is a non-zero positive real number, and since this measure is countably additive, it follows that G cannot be countable unless it is finite.

8. Finitely generated group towers.

Definition. A group tower $\pi = \{G_s, \alpha_s^t, s \in S\}$ will be said to be finitely generated if there exists an integer n such that each G_s is generated by n elements; we shall then say that π is generated by n generators. A family $\{(s_1, s_2, \cdots, s_n), s \in S\}$ will be said to be *a consistent family of n-generators of π* if for each s in S, $(s_1, s_2, \cdots, s_n)$ are generators of G_s and for each $s < t$ in S, $\alpha_s^t(t_j) = s_j$ for $j = 1, 2, \cdots, n$. Note that if a cofinal subset of π has a consistent family of n generators, then it can uniquely be extended to a consistent family of n generators of π. If π has a consistent family of n generators, then π *will be said to be consistently generated by n generators.*

LEMMA 18. *Let G be a finitely generated group and let m be a given integer. Then G contains only a finite number of subgroups of index m.*

Proof. Since any subgroup of G of index m contains a normal subgroup of G of index which is a factor of $m!$ ($=$ the order of the symmetric group on m letters),[7] it is enough to prove our assertion for normal subgroups of G. Let $x_1, x_2, \cdots, x_n$ be a set of generators of G. Let S_m be the symmetric group on m letters. A homomorphism of G into S_m is uniquely determined by giving the images of $x_1, x_2, \cdots, x_n$, and for each x_j, there are at most $m!$ possible choices, hence the number q of distinct homomorphisms of G into S_m is finite, namely, $q \leqq (m!)^n$. For a normal subgroup N of G of index m, let f_N denote the canonical homomorphism of G onto G/N. Since any group of order m is isomorphic to a subgroup of S_m, we can fix an isomorphism g_N of G/N into S_m; set $h_N = g_N f_N$. Then h_N is a homomorphism of G into S_m and $h_N^{-1}(1) = N$. Therefore distinct normal subgroups N of G of index m give rise to distinct homomorphisms of G into S_n. Hence the number of such subgroups is $\leqq q$ and hence finite.

LEMMA 19. *Let $\pi = \{G_s, \alpha_s{}^t, s \in S\}$ be a finitely generated group tower. Then for any given integer m, there are only a finite number of G_s of order m.*

Proof. Suppose π is generated by n generators and let F_n be the free group on m generators. By Lemma 18, there are only a finite number q of normal subgroups of F_n of index m. Suppose, if possible, that there are more than q groups in π of order m, choose $q+1$ of them, say $G_{s1}, G_{s2}, \cdots, G_{sq+1}$. Since S is directed, there exists s in S with $s > sj$ for $j = 1, 2, \cdots, q+1$. Let f be a homomorphism of F_n onto G_s and let $f_j = \alpha_{sj}{}^s f$. Since π is a tower, the kernels of $\alpha_{s1}{}^s, \alpha_{s2}{}^s, \cdots, \alpha_{sq+1}{}^s$ are all distinct, and hence the kernels of $f_1, f_2, \cdots, f_{q+1}$ are all distinct; since these are all normal subgroups of F_n of index m, this is a contradiction.

LEMMA 20. *Let S be a countable directed set. Then S contains an ascending cofinal sequence $s_1 < s_2 < \cdots$.*

Proof. If S is finite,it is enough to take $s_1 = s_2 = \cdots =$ *the* maximum of S. Now assume S is not finite and let $n_1, n_2, \cdots$ be a counting up of S. Let $s_1 = u_1$, choose s_2 with $s_2 > s_1$ and $s_2 > u_2$, choose s_3 with $s_3 > s_2$ and $s_3 > u_3, \cdots$, etc.

LEMMA 21. *Let $\pi = \{G_s, s \in S\}$ be a group tower. Assume that there are only a finite number of groups G_s of any given order. Then S contains a cofinal ascending sequence.*

Proof. The assumption implies that S is countable. Now apply Lemma 20.

PROPOSITION 3. *Let $\pi = \{G_s, s \in S\}$ be a finitely generated group tower. Then S contains a cofinal ascending sequence.*

Proof. Follows from Lemmas 19 and 21.

LEMMA 22. *Let $\pi = \{G_s, \alpha_s{}^t, s \in S\}$ be a group tower. Then the following three conditions are equivalent:* (1) *π is consistently generated by n generators*; (2) *π has a weak parent group generated by n generators*; (3) *π is isomorphic to a subtower of the derived group tower of a group generated by n generators. Furthermore, if $\{(s_1, s_2, \cdots, s_n), s \in S\}$ is a consistent family of n generators of π, then there exists a group G generated by n generators $a_1, a_2, \cdots, a_n$, and a weak parent map $g = \{g_s\colon G \to G_s, s \in S\}$ of G onto π such that $g_s(a_j) = s_j$ for all s in S and $j = 1, 2, \cdots, n$; G is unique in the sense that if H is any other group generated by n generators $b_1, b_2, \cdots, b_n$ with a weak parent map $h = \{h_s\colon H \to G_s, s \in S\}$ such that $h_s(b_j) = s_j$ for all s in S and $j = 1, 2, \cdots, n$, then there exists a* (*unique*) *isomorphism of G onto H with $a_j \to b_j$ for $j = 1, 2, \cdots, n$.*

Proof. It is obvious that (2) implies (3); also (3) implies that there exists a group P with n generators and a family p of homomorphisms of P onto the various G_s consistent with the maps $\alpha_s{}^t$; if Q is the intersection of the kernels of all the members of p, then P/Q is a weak parent group of π and is generated by n generators, which implies (2); again it is obvious that (3) implies (1). Now assume (1) and let $\{(s_1, s_2, \cdots, s_n), s \in S\}$ be a consistent family of n generators of π. Let F_n be the free group on n generators $x_1, x_2, \cdots x_n$, and let f_s be the homomorphism of F_n onto G_s given by $f_s(x_j) = s_j$. Let $M = \bigcap_{s \in S} f_s^{-1}(1)$, let u be the canonical homomorphism of F_n onto $G = F_n/M$ and let $a_j = u(x_j)$. Since $M \subset f_s^{-1}(1)$, there exists a unique homomorphism g_s of G onto G_s with $f_s = g_s u$. Then $g = \{g_s, s \in S\}$ is a weak parent map of G onto π with $g_s(a_j) = s_j$ for s in S and $j = 1, 2, \cdots, n$. Now let $H, b_1, b_2, \cdots, b_n, h$ be as stated. Since F_n is free, there exists a unique homomorphism v of F_n onto H with $v(x_j) = b_j$ for $j = 1, 2, \cdots, n$. Then $h_s v(x_j) = s_j$, and hence $h_s v = f_s$ for all s in S, and hence $v^{-1}(1) \bigcap_{s \in S} f_s^{-1}(1)$. Also $d \in F_n$, $d \notin u^{-1}(1)$ implies that $v(d) \neq 1$, which, in view of the assumption that h is a weak parent map, implies that $f_s(d) = h_s(v(d)) \neq 1$, i.e., $d \notin f_s^{-1}(1)$. Hence $v^{-1}(1) \supset \bigcap_{s \in S} f_s^{-1}(1)$. Therefore $v^{-1}(1) = \bigcap_{s \in S} f_s^{-1}(1) = M$, and hence vu^{-1} is an isomorphism of G onto H with $a_j \to b_j$ for $j = 1, 2, \cdots, n$.

Now let G be a group generated by t generators $a_1, a_2, \cdots, a_t$ such that a_1 generates a normal subgroup A_1 in $G_1 = G$, a_2 generates a normal subgroup A_2 in $G_2 = G_1/A_1$, a_3 generates a normal subgroup A_3 in $G_3 = G_2/A_2$, $\cdots, a_{t-1}$ generates a normal subgroup A_{t-1} in $G_{t-1} = G_{t-2}/A_{t-2}$, and a_t generates $A_t = G_t = G_{t-1}/A_{t-1}$.

For a normal subgroup H of G, let f_H denote the canonical homomorphism of G onto $G_H = G/H$. Then it is clear that G_H has the same property as G with respect to th sequence $f_H(a_1), f_H(a_2), \cdots, f_H(a_t)$; also it is clear that the groups which now correspond to $G_1, G_2, \cdots, G_t, A_1, A_2, \cdots, A_t$ are the canonical images of these under f_H, i.e., in a natural way, they are $f_H(G_1)$, $f_H(G_2), \cdots, f_H(G_t)$, $f_H(A_1), f_H(A_2), \cdots, f_H(A_t)$. Note that *if the order of one of the a_j in G_j is finite, then it is divisible by the order of $f_H(a_j)$ in $f_H(G_j)$.* Also observe that *the order of $f_H(a_1)$ in G_H is simply the order of $A_1/(H \cap A_1)$, i.e., the order of a_1 in $A_1/(H \cap A_1)$.*

LEMMA 23. *Let $n_1, n_2, \cdots, n_t$ be given positive integers. Let S be the set of normal subgroups H of G such tat in $f_H(G_j)$ the order of $f_H(a_j)$ divides n_j for $j = 1, 2, \cdots, t$. Then for any subset S' of S, $\bigcap_{H \in S'} H$ is again in S.*

Proof. First observe that S is not empty since it contains G itself Let $K = \bigcap_{H \in S} H$. Now if K is in S, then any normal subgroup of G containing S is obviously again in S, and since the intersection of the members of any subset of S contains K, that intersection must lie in S. Hence it is enough to prove that K is in S. Now we shall make induction on t. For $t = 1$, G is cyclic, and hence K is the *unique* normal subgroup of G whose index is n_1 if G is infinite, and the greatest common divisor of n_1 and the order of G if G is finite. Next assume that $t > 1$ and that the lemma is true for $t - 1$. We have $K \cap A_1 = \bigcap_{H \in S} (H \cap A_1)$, and hence, applying the lemma to A_1, we conclude by the last italicized statement before the lemma that the order of a_1 in $f_K(G_1)$ is a factor of n_1. Next, let ϕ be the canonical homomorphism of $G = G_1$ onto $G_2 = G_1/A_1$. Then H in S implies that $\phi(H)$ satisfies the conditions of the lemma for G_2 with respect to $\phi(a_2), \phi(a_3), \cdots, \phi(a_t)$ and $n_2, n_3, \cdots, n_t$ respectively. Hence, applying the lemma for $t - 1$ to this and observing that $\phi(K) = \bigcap_{H \in S} \phi(H)$, we conclude that the orders of $f_K(a_2)$, $\cdots, f_K(a_t)$ in $f_K(G_2), \cdots, f_K(G_t)$ divide $n_2, \cdots, n_t$ respectively. Therefore K is in S.

LEMMA 24. (i) *Let $n_1, \cdots, n_t$ be given positive integers. Then there exists at most one normal subgroup K in G such that $f_K(a_j)$ is of order n_j*

in $f_K(G_j)$ for $j = 1, \cdots, t$. (ii) *Now let $m_1, \cdots, m_t$ be another set of positive integers such that m_j divides n_j for $j = 1, \cdots, t$. Assume that there exist normal subgroups H and K of G such that $f_H(a_j)$ and $f_K(a_j)$ are, respectively, of orders m_j and n_j in $f_H(G_j)$ and $f_K(G_j)$ for $j = 1, \cdots, t$. Then $H \supset K$.*

Proof. (i) Let K and K_1 be two normal subgroups with the required property and let $K^* = K \cap K_1$. Then by Lemma 23, the order n^*_j of $f_{K^*}(a_j)$ in $f_{K^*}(G_j)$ divides n_j. Since $K^* \subset K$, $f_K(G)$ is canonically homomorphic to $f_{K^*}(G)$, and hence, by the last but one italicized statement before Lemma 23, n^*_j divides n_j. Therefore $n^*_j = n_j$. Since $G/K = f_K(G_1) \supset f_K(G_2) \supset \cdots \supset f_K(G_t) \supset 1$ is a normal sequence of G/K in which the orders of the successive factors are $n_1, \cdots, n_2$, the order of G/K must be $n_1 n_2 \cdots n_t$. Similarly, the order of G/K_1 is $n_1 n_2 \cdots n_t$ and the order of G/K^* is $n^*_1 n^*_2 \cdots n^*_t = n_1 n_2 \cdots n_t$. Since $K^* \subset K$ and $K^* \subset K_1$, this says that $K = K^* = K_1$.

(ii) Let $L = H \cap K$. Then by an argumnet similar to the one used in the proof of (i), we conclude that the order of G/L equals the order of G/K, and hence $L = K$, i.e., $H \supset K$.

9. Solvable, nilpotent and abelian group towers. [References: K1, Chapter IV and K2, Chapters XIV and XV.] Let G be a group. Recall that G is *solvable* means that G has a finite solvable normal series and G is *nilpotent* means that G has a finite central series. We shall say that G is *m-step* solvable if G has a solvable normal series of length m, and we shall say that G is *m-step nilpotent* if G has a central series of length m. In this section, for any group G, we shall denote by $D_q(G)$ and $E_q(G)$ subgroups of G defined by setting: $D_0(G) = E_0(G) = G$, $D_j(G) =$ the commutator subgroup of $D_{j-1}(G)$, $E_j(G) =$ the subgroup of G generated by the commutators of G and $E_{j-1}(G)$. It is well known that G is m-step solvable if and only if $D_m(G) = 1$ and that G is m-step nilpotent if and only if $E_m(G) = 1$; in particular, G is 1-step nilpotent if and only if G is abelian and G is 2-step nilpotent if and only if the commutator subgroup $D_1(G)$ of G is contained in the center of G. *We shall say that a group tower π is m-step solvable (respectively: m-step nilpotent, abelian) if and only if every group in π is m-step solvable (respectively: m-step nilpotent, abelian).*

LEMMA 25. *Let $f\colon G \to H$ be an onto homomorphism.* i *Then for all j, we have $f(D_j(G)) = D_j(H)$ and $f(E_j(G)) = E_j(H)$.*

Proof. It is obvious for D_0 and E_0; suppose $j > 1$ and assume it is true for D_{j-1} and E_{j-1}. Now u in $D_j(G)$ implies that there exist a, b in $D_{j-1}(G)$

with $u = aba^{-1}b^{-1}$; then $f(u) = f(a)f(b)f(a)^{-1}f(b)^{-1}$ and this is in $D_j(H)$ since, by assumption, $f(a)$ and $f(b)$ are in $D_{j-1}(H)$; conversely, u^* in $D_j(H)$ implies that there exist a^*, b^* in $D_{j-1}(H)$ with $u = a^*b^*a^{*-1}b^{*-1}$; then, by assumption, there exist a, b in $D_{j-1}(G)$ with $f(a) = a^*$ and $f(b) = b^*$, hence $u = aba^{-1}b^{-1}$ is in $D_j(G)$ and $f(u) = u^*$. Again, u in $E_j(G)$ implies that there exist a in $E_{j-1}(G)$ and b in G with either $u = aba^{-1}b^{-1}$ or $u = bab^{-1}a^{-1}$; then, by assumption, $f(a)$ is in $E_{j-1}(H)$ and $f(u) = f(a)f(b)f(a)^{-1}f(b)^{-1}$ or $f(u) = f(b)f(a)f(b)^{-1}f(a)^{-1}$ respectively, so that, in either case, $f(u)$ is in $E_j(H)$; conversely, u^* in $E_j(H)$ implies $\cdots$, etc. The proof is complete by induction.

LEMMA 26. *Let G be a group and let S be a set of onto homomorphisms $s\colon G \to H_s$ of G such that $\bigcap_{s \in S} s^{-1}(1) = 1$. If there exists an integer m such that for all s in S, H_s is m-step solvable (respectively m-step nilpotent), then G is m-step solvable (respectively, m-step nilpotent). For a and b in G, if $s(a)$ and $s(b)$ commute for all s in S, then a and b commute; in particular, if H_s is abelian for all s in S, then G is abelian.*

Proof. Assume that H_s is m-step solvable for all s in S. Then by Lemma 25, for each s in S, we have $D_m(G) \subset s^{-1}(D_m(H_s)) = s^{-1}(1)$. Therefore $D_m(G) \subset \bigcap_{s \in S} s^{-1}(1) = 1$. Hence $D_m(G) = 1$, i.e., G is m-step solvable. The statement for 'm-step nilpotent' follows similarly. For a, b in G, if $s(a)$ and $s(b)$ commute for each s in S, then

$$aba^{-1}b^{-1} \in \bigcap_{s \in S} s^{-1}(1) = \{1\},$$

and hence a and b commute.

Specializing Lemma 26 to group towers, we may state:

PROPOSITION 4. *Let G be a weak parent group of a group tower π. Then G is m-step solvable (respectively: m-step nilpotent, abelian) if and only if π is m-step solvable (respectively: m-step nilpotent, abelian) if and only if every weak parent group of π is m-step solvable (respectively: m-step nilpotent, abelian). For a, b in G, a and b commute if and only if their images (under a given weak parent map of G onto π) in every member of π commute.*

LEMMA 27. *In a finitely generated abelian G, the intersection of subgroups of finite index is 1, i.e., G is a parent group of its derived group tower.*

Proof. G is a direct product of cyclic subgroups $G_1, G_2, \cdots, G_n$; *let f_j* be the projection of G onto G_j. We want to show that $g \in G$, $g \neq 1$ implies

that there exists a homomorphism ϕ of G onto a finite group such that g is not mapped onto the identity. Now $g \neq 1$ implies that $f_j(g) \neq 1$ for some j, hence we may replace G by G_j, i.e., we may assume that G is cyclic. If G is finite, we may take ϕ to be the identity isomorphism of G onto G. If G is infinite, let a be a generator of G and let $g = a^u$, let n be a positive integer which does not divide u, let H be a cyclic group of order n with generator b and define ϕ by taking $\phi(a) = b$.

LEMMA 28. *Let G be a group which is generated by t subgroups $H_1, H_2, \cdots, H_t$ such that H_j is abelian and is normal in G. Then G is t-step nilpotent.*

Proof. First observe that if A and B are two normal subgroups in a group C and C is generated by A and B, then denoting the centers of A, B, C by $Z(A)$, $Z(B)$, $Z(C)$ respectively, we have that $Z(A) \cap Z(B) \subset Z(C)$. For the assumption implies that $C = AB$, i.e., every element $c \in C$ is of the form $c = ab$ with $a \in A$ and $b \in B$. Now $u \in Z(A) \cap Z(B)$ implies that $uc = uab = aub = abu = cu$, i.e., $u \in Z(C)$.

The above statement together with a trivial induction shows that the center $Z(G)$ of G contains the intersection of the centers of $H_1, H_2, \cdots, H_t$; since each H_j is abelian, it is its own center and hence $Z(G) \supset H_1 \cap H_2 \cap \cdots \cap H_t$.

Now we shall prove the lemma by induction on t. For $t = 1$, G is abelian and hence 1-step nilpotent; now suppose $t > 1$ and assume the lemma for $t - 1$. Let f_j be the canonical homomorphism of G onto G/H_j. Since G/H_j is generated by the $t - 1$ subgroups $H_k/(H_j \cap H_k)$ $(k \neq j)$ each of which is abelian and normal in G/H_j, by the induction hypothesis, we conclude that $E_{t-1}(G/H_j) = 1$. By Lemma 25, we have

$$E_{t-1}(G) \subset \bigcap_{j=1}^{t} f_j^{-1}(E_{t-1}(G/H_j)) = \bigcap_{j=1}^{t} f_j^{-1}(1) = \bigcap_{j=1}^{t} H_j \subset Z(G).$$

Hence $E_t(G) = 1$.

10. Algebraic fundamental groups. Let K be an n dimensional algebraic function field over an algebraically closed ground field k of characteristic p and let Λ be a fixed alegbraic closure of K. Let V be a normal projective model of K/k and let W be a proper subvariety of V. Then as in Section 4 of [A3], we define:

[8] The notions here will be much more refined than those introduced in Section 4 of [A3]; note that fixing an algebraic closure Λ does not affect the notions of that section.

$\Omega(V-W)=$ the family of finite separable algebraic extensions L of K in Λ for which $\Delta(L/V)\subset W$.

$\Omega_g(V-W)=$ the family of members L of $\Omega(V-W)$ such that L/K is galois.

$\Omega'_g(V-W)=$ the family of members L of $\Omega_g(V-W)$ such that L/V is tamely ramified.

$\Omega^*_g(V-W)=$ the family of members L of $\Omega_g(V-W)$ such that $[L:K]\not\equiv 0 \pmod p$ in case $p\neq 0$ (and no restriction if $p=0$).

The galois groups over K of the members of $\Omega_g(V-W)$, or $\Omega'_g(V-W)$, or $\Omega^*_g(V-W)$ form, under the natural homomorphisms, group towers (Lemma 12, Section 2). We set

$\pi(V-W)=$ *the fundamental group tower of* $V-W$
$=$ the group tower of galois groups over K members of $\Omega_g(V-W)$.

$\pi'(V-W)=$ the *tame fundamental group tower* of $V-W$
$=$ the group tower of galois groups over K members of $\Omega'_g(V-W)$.

$\pi^*(V-W)=$ the *reduced fundamental group tower of* $V-W$
$=$ the group tower of galois groups over K members of $\Omega^*_g(V-W)$.

For brevity, sometimes we shall omit the adjective 'fundamental' from the above group towers.[9] Note that $\pi^*(V-W)$ is a subtower of $\pi'(V-W)$ which itself is a subtower of $\pi(V-W)$; if $p\neq 0$, then $\pi^*(V-W)$ consists exactly of those groups of $\pi(V-W)$ whose orders are prime to p, while if $p=0$, then $\pi^*(V-W)=\pi'(V-W)=\pi(V-W)$.

We shall say that $V-W$ is respectively: (1) *simply connected,* (2) *tamely simply connected,* or (3) *reduced simply connected*; according as: (1*) $\pi(V-W)=1$ (i.e., consists of the trivial group alone), (2*) $\pi'(V-W)=1$, or (3*) $\pi^*(V-W)=1$. It follows (Lemmas 5 and 7 of Section 2) that $V-W$ is simply connected (respectively: tamely simply connected) if and only if there does not exist any finite separable algebraic extension K^* of K ($K^*\neq K$, K^*/K not necessarily galois) for which $\Delta(K^*/K)\subset W$ (respec-

[9] Changing the algebraic closure Λ of K will affect these group towers only up to canonical isomorphisms.

tively: $\Delta(K^*/K) \subset W$ and K^*/V tamely ramified). We note the following trivial lemma.

LEMMA 29. (i) *$\pi(V-W)=1$, $\pi'(V-W)=1$, $\pi^*(V-W)$ respectively imply that $\pi(V)=1$, $\pi'(V)=1$, $\pi^*(V)=1$.* (ii) *$\pi(V)=\pi'(V)$ so that, in particular, V is simply connected if and only if V is tamely simply connected if and only if there does not exist any finite separable algebraic extension K^*/K ($K^*\neq K$, K^*/K not necessarily galois) such that K^*/V is unramified.*

Also we have the following:

LEMMA 30. *If V_1 and V_2 are two nonsingular projective models of K/k, then $\pi(V_1)=\pi(V_2)$, i.e., the fundamental group tower is a birational invariant for nonsingular projective models.*

Proof. Let K^*/K be a galois extension. Lemmas 15 and 16 of Section 2 imply that K^*/V is unramified if and only if each valuation of K/k is unramified in K^*.

LEMMA 31. *If V_1 and V_2 are two nonsingular projective models of K/k, then $\pi^*(V_1)=\pi^*(V_2)$ [so that $\pi(V_1)=\pi(V_2)$ in case $p=0$], i.e., the reduced fundamental group tower (and hence the fundamental group tower in case $p=0$) is a birational invariant for nonsingular projective models.*

Proof. This is a corollary of Lemma 30 and also follows from Lemmas 15 and 17 (Section 2).

Remark 5. Note that in the classical case when the ground field is the field of complex numbers, a stronger assertion can be made, namely, it is well known that the topological fundamental group is a birational invariant for nonsingular projective models.

Let G be a group. Then G will, respectively, be said to be (1) *an unrestricted fundamental weak parent group of $V-W$,* (1′) *an unrestricted tame fundamental weak parent group of $V-W$,* (1*) *an unrestricted reduced fundamental weak parent group of $V-W$* if G is a weak parent group, respectively, of $\pi(V-W)$, $\pi'(V-W)$, $\pi^*(V-W)$. Secondly, G will, respectively, be said to be (2) *an unrestricted fundamental parent group of $V-W$,* (2′) *an unrestricted tame fundamental parent group of $V-W$,* (2*) *an unrestricted reduced fundamental parent* group of $V-W$ if G is a parent group, respectively, of $\pi(V-W)$, $\pi'(V-W)$, $\pi^*(V-W)$. Thirdly, G will, respectively, be said to be (3′) *a tame fundamental weak parent group*

of $V-W$, and (3^*) *a reduced fundamental weak parent group* of $V-W$ if G is *finitely generated* and if G is a weak parent group, respectively, of $\pi'(V-W)$ and $\pi^*(V-W)$.

Next, G will be said to be ($4'$) *a tame fundamental parent* group of $V-W$ if G is *finitely generated* and G is a parent group of $\pi(V-W)$ $\pi'(V-W)=\pi^*(V-W)$ in case $p=0$, while G is a modulo p parent group of $\pi'(V-W)$ in case $p\neq 0$. Finally, G will be said to be (4^*) *a reduced fundamental parent group of* $V-W$ if G is *finitely generated* and G is a parent group of $\pi(V-W)=\pi'(V-W)=\pi^*(V-W)$ in case $p=0$, while G is a modulo p parent group of $\pi^*(V-W)$ in case $p\neq 0$.

The adjectives 'fundamental' and 'group' from all the above objects may be omitted for brevity. Observe that for $p=0$, the adjectives 'tame' and 'reduced' are superfluous.

Remark 6. Proposition 4 of Section 9 tells us that if one restricted tame fundamental weak parent group of $V-M$ is t-step solvable (respectively: t-step nilpotent, abelian), then so is any other unrestricted tame fundamental weak parent group of $V-W$ and hence, any tame fundamental parent group of $V-W$, etc.

Remark 7. The galois groups over K of the compositums, respectively, of $\Omega_g(V-W)$, $\Omega'_g(V-W)$ and $\Omega^*_g(V-W)$ are naturally isomorphic to the inverse limits, respectively, of $\pi(V-W)$, $\pi'(V-W)$ and $\pi^*(V-W)$, and hence they are, respectively, an unrestricted fundamental weak parent group of $V-W$, an unrestricted tame fundamental weak parent group of $V-W$ and an unrestricted reduced fundamental weak parent group of $V-W$. In connection with the existence of unrestricted parent groups, see Example 1 and Remark 4 of Section 7. For further discussion of the definitions of this section, see Section 16.

C. Main Results.

Throughout the rest of the paper, k will denote an algebraically closed field of characteristic p.

11. Finite coverings. Let V be a nonsingular projective n dimensional algebraic variety over k, let $K=k(V)$, let K^* be a finite separable algebraic extension of K, let V^* be a K^*-normalization of V, and let ϕ be the rational map of V^* onto V.

PROPOSITION 5. *If W is an irreducible $n-1$ dimensional subvariety of V such that* $\dim |W| > 1$, *then $\phi^{-1}(W)$ is connected.*

Proof. Since $\dim |W| > 1$ and W is irreducible, it follows by the generalized theorem of Bertini (Section 4) that $|W|$ is not composite with a pencil. Therefore $\phi^{-1}(|W|)$ is not composite with a pencil, and hence, again by the generalized theorem of Bertini, a 'general' member of $\phi^{-1}(W)$ is irreducible. Therefore by Zariski's degeneration principle [Z5, see also C1], (the support of) every member of $\phi^{-1}(|W|)$ is connected. Since $\phi^{-1}(W)$ is the support of the divisor in $\phi^{-1}(|W|)$ corresponding to W, we conclude that $\phi^{-1}(W)$ is connected.

PROPOSITION 6. *Now suppose that K^*/V is tamely ramified. Let W be a pure $n-1$ dimensional subvariety of V, with $W_1, W_2, \cdots, W_t$ as its distinct irreducible components, such that $\Delta(K^*/V) \subset W$. Assume that:* (1) $\dim |W_j| > 1$ *for $j = 1, 2, \cdots, t$; and* (2) *W has only normal crossings.*[10] *Then $\phi^{-1}(W_j)$ is irreducible for $j = 1, 2, \cdots, t$.*

Proof. Let K' be a least galois extension of K containing K^*; then K'/V is tamely ramified and $\Delta(K'/V) \subset W$ (Lemmas 5 and 7, Section 2), and if the 'inverse image' of W_j on the K'-normalization were irreducible, then $\phi^{-1}(W_j)$ would *a fortiori* would be irreducible; therefore we may assume that K^*/K is galois to begin with. By Proposition 5, $\phi^{-1}(W_j)$ is connected. Suppose, if possible, that $\phi^{-1}(W_j)$ is reducible; then at least two distinct irreducible components of $\phi^{-1}(W_j)$ must have a point P^* in common. Let $P = \phi(P^*)$. Now W_j is an irreducible component of $\Delta(K^*/V)$ and W has a normal crossing at P implies that $\Delta(K^*/V)$ has a normal crossing at P; therefore, by Proposition 2 (Section 4), only one irreducible component of $\phi^{-1}(W_j)$ passes through P^*. This is a contradiction, and hence the proposition is proved.

THEOREM 1. *Let K be an n dimensional algebraic function field over an algebraically closed ground field k of characteristic p, let V be a non-singular projective model of K/k, let W be a pure $n-1$ dimensional subvariety of V with distinct irreducible components $W_1, W_2, \cdots, W_t$. Let K^*/K be a galois extension such that K^*/V is tamely ramified and $\Delta(K^*/V) \subset W$. Let V^* be a K^*-normalization of V and let ϕ be the rational map of V^* onto V.*
Assume that:

[10] I. e., W has a normal crossing at each of its points.

(1) $\dim |W_j| > 1$ *for* $j = 1, 2, \cdots, t$;

(2) W *has only normal crossings; and*

(3) V *is simply connected.*

Then:

(A) $W^*_j = \phi^{-1}(W_j)$ *is irreducible for* $j = 1, 2, \cdots, t$.

(B) *The inertia group* $G_i(W^*_j/W_j)$ *is a cyclic normal subgroup (of order prime to* p *in case* $p \neq 0$) *of* $G(K^*/K)$ *for* $j = 1, 2, \cdots, t$. *Let* a_j *be a generator of* $G_i(W^*_j/W_j)$.

(C) $G(K^*/K)$ *is generated by* $G_i(W^*_1/W_1), G_i(W^*_2/W_2), \cdots, G_i(W^*_t/W_t)$, *Hence*

(D) $G(K^*/K)$ *is generated by the* t *generators* $a_1, a_2, \cdots, a_t$ *each of which generates a normal subgroup.*

(E) $G(K^*/K)$ *is* t*-step nilpotent and its order is not divisible by* p *in case* $p \neq 0$.

(F) *If* W_j *and* W_k *have a point in common, then* a_j *and* a_k *commute in* $G(K^*/K)$.

(G) *If* $W_1, W_2, \cdots, W_t$ *are pairwise connected, i.e., any two have a point in common, then* $G(K^*/K)$ *is abelian.*

Proof. (A) follows from Proposition 6. Hence we can talk of $G_i(W^*_j/W_j)$; since $Q(W_j, V)$ does not split in K^*, $G_s(W^*_j/W_j) = G(K^*/K)$, and hence (B) follows from Lemmas 1 and 14 of Section 2. Now let H be the subgroup of $G(K^*/K)$ generated by $a_1, a_2, \cdots, a_t$ and let K_1 be the fixed field of H. Then $W_1, W_2, \cdots, W_t$ are unramified in H (Lemma 13, Section 2) and K_1/V is tamely ramified (Lemma 12, Section 2), and hence K_1/V is unramified (Lemma 17, Section 2). Therefore assumption (3) implies that $K_1 = K$, i.e., $H = G(K^*/K)$, which gives (C). (D) is only a rephrasing of (C). (E) follows from (D) and Lemma 28 of Section 9.

Now assume that W_j and W_k have a point P in common and let P^* be a point in $\phi^{-1}(P)$. Then $P^* \in W^*_j$ and $P^* \in W^*_k$ so that $G_i(W^*_j/W_j) \subset G_i(P^*/P)$ and $G_i(W^*_k/W_k) \subset G_i(P^*/P)$ (Lemma 10, Section 2), i.e., a_j and a_k are in $G_i(P^*/P)$. Now assumption (2) implies that P is a normal crossing of $\Delta(K^*/V)$, and hence, by Proposition 1 (Section 2), $G_i(P^*/P)$ is abelian; therefore a_j and a_k commute, which proves (F). Finally, (G) follows from (D) and (F).

12. Fundamental weak parent groups.

THEOREM 2. *Let K be an n dimensional algebraic function field over an algebraically closed ground field k, let V be a nonsingular projective model of K/k, and let W be a pure $n-1$ dimensional subvariety of V with irreducible components $W_1, W_2, \cdots, W_t$. Assume that:*

(1) $\dim |W_j| > 1$ *for* $j = 1, 2, \cdots, t$;

(2) *W has only normal crossings; and*

(3) *V is simply connected.*

Then:

(A) *$V - W$ has a tame fundamental weak parent group G generated by t generators $a_1, a_2, \cdots, a_t$ with a weak parent map f of G onto $\pi'(V-W)$ such that, in each member H of $\pi'(V-W)$, a_j (i.e., the f image of a_j) generates the cyclic (and normal in H) inertia group over W_j of the unique irreducible subvariety corresponding to W_j on a normalization of V in the galois extension of K corresponding to H; also, a_j and a_k commute in G if W_j and W_k have a point in common.*

(B) *Every unrestricted tame fundamental weak parent group*[11] *of $V-W$ is t-step nilpotent.*

(C) *If $W_1, W_2, \cdots, W_t$ are pairwise connected, then every unrestricted tame fundamental weak parent group*[11] *of $V - W$ is abelian.*

(D) $\pi^*(V-W) = \pi'(V-W)$.

Proof. First we assert that (α): *$\pi'(V-W)$ contains an ascending cofinal sequence*; we shall give several proofs of this. (i) By Theorem 1, $\pi'(V-W)$ is finitely generated, and hence (α) follows from Proposition 3 of Section 8.[12] (ii) For any K^* as in Theorem 1, $G = G(K^*/K)$ and the generators $a_1, a_2, \cdots, a_2$ satisfy the description given before Lemma 23 of Section 8; using the notation of that description, we can set $n_j(K^*)$ = order of a_j in G_j; note that $n_j(K^*)$ does not depend on the particular generator a_j of $G_i(W^*_j : W_j)$.[13] Then in view of the fact that the inertia groups project

[11] In particular, G and the inverse limit of $\pi'(V-W)$, i.e., the galois group over K of the compositum of $\Omega'_g(V-W)$, as well as every unrestricted tame fundamental parent group of $V-W$.

[12] Proposition 3 is based on the general group theoretic Lemmas 18 and 19 and also on Lemmas 20 and 21, whereas Lemma 24 is based on Lemma 23 dealing with 'nice' groups, and so the proof of Lemma 24 is less sophisticated than that of Proposition 3.

[13] G will stand for $G(K^*/K)$ only in this sentence and should not be confused with

properly (Lemma 2 of Section 2), taking compositums and applying Lemma 24 of Section 8, we conclude the following:[12] Given galois extensions K_1 and K_2 of K which are tamely ramified over V and for which $\Delta(K_1/V)$ and $\Delta(K_2/V)$ are contained in W, $n_j(K_1)=n_j(K_2)$ for $j=1,2,\cdots,t$ implies that $K_1=K_2$, and $n_j(K_1)$ divides $n_j(K_2)$ for $j=1,2,\cdots,t$ implies that $K_1\subset K_2$. From this, it follows that $\pi'(V-W)$ is countable; and hence we can conclude (α) from Lemma 20 of Section 8, or we may proceed thus: Let $m_1,m_2,\cdots$ be a sequence of positive integers such that m_q divides m_{q+1} for all q and any positive integer divides some m_q; for instance, take $m_q=q\,!$. By Lemma 24, there exists a unique (finite) galois extension K_q (in a fixed algebraic closure of K) tamely ramified over V, with $\Delta(K_q/V)\subset W$, for which $n_j(K_q)$ divides m_q for $j=1,\cdots,t$ and such that K_q contains every other finite galois extension of K with these properties. Again by Lemma 24, it follows that $G(K_1/K)<G(K_2/K)<\cdots<G(K_q/K)<G(K_{q+1}/K)<\cdots$ is a cofinal ascending sequence in $\pi'(V-W)$. (iii) Since there are only a finite number of tamely ramified coverings of V of a given degree with branch loci contained in W (Remark 9 of Section 6 of [A3]),[4] (α) follows from the trivial Lemma 21 of Section 8.

Now let $1=G^1<G^2<\cdots$ be an ascending cofinal sequence in $\pi'(V-W)$, let $K=K^1\subset K^2\subset\cdots$ be the corresponding galois extensions of K with $G(K^q/K)=G^q$, and let ϕ_q be the rational map of a K^q-normalization of V onto V. By Theorem 1, $\phi_q^{-1}(W_j)$ is irreducible; let

$$G^q{}_j=G_i(\phi_q^{-1}(W_j)/W_j).$$

Theorem 1 tells us that $G^q{}_1,G^q{}_2,\cdots,G^q{}_t$ are cyclic normal subgroups of G^q and they generate G^q. We shall show that generators $b^q{}_j$ of $G^q{}_j$ can be so chosen that for all q, we have $u_{q+1}(b^{q+1}{}_j)=b^q{}_j$ for $j=1,2,\cdots,t$, where u_{q+1} is the canonical homomorphism of G^{q+1} onto G^q. For $q=1$, we of course have $b^1{}_1=b^1{}_2=\cdots=b^1{}_t=1$; now suppose $q>1$ and that $b^h{}_j$ have been so chosen for all $h<q$. Lemma 2 of Section 2 tells us that $u_{q+1}^{-1}(G^q{}_j)=G^{q+1}{}_j$, and hence $u_{q+1}^{-1}(b^q{}_j)$ is a generator of $G^{q+1}{}_j$ and we set $b^{q+1}{}_j=u_{q+1}^{-1}(b^q{}_j)$. Since $G^1<G^2<\cdots$ is cofinal in $\pi'(V-W)$, invoking Lemma 22 of Section 8, we can find a group G on t generators $a_1,a_2,\cdots,a_t$ and a weak parent map f of G onto $\pi'(V-W)$ such that the f image of a_j in G^q is $b^q{}_j$ for all q and $j=1,\cdots,t$. Again since $G^1<G^2<\cdots$ is cofinial in $\pi'(V-W)$, it follows from Lemma 2 of Section 2, conclusion (F) of Theorem 1 and

the G in the conclusion (A) of Theorem 2. Also only in this sentence, the notation of Theorem 1 is used, for instance $W^*{}_j$, etc.

Proposition 4 of Section 9 that G satisfies the description of conclusion (A) of Theorem 2 with respect to the set of generators $a_1, a_2, \cdots, a_t$ and the weak parent map f. (B) and (C) follow from (A) and conclusions (E) and (G) of Theorem 1 by invoking Proposition 4 of Section 9. (D) is given in conclusion (B) of Theorem 1. Q. E. D.

Remark 8. Referring to the conclusion (A) of Theorem 2, Lemma 22 of Section 8 tells us that G is unique in the following sense. If G' is a group on t generators $a'_1, a'_2, \cdots, a'_t$ with a weak parent map f' of G' onto $\pi'(V-W)$ such that for $j=1,2,\cdots,t$, and for each H in $\pi'(V-W)$, the f and f' images respectively of a_j, a'_j coincide, i. e., a_j and a'_j are mapped onto the *same* generator of the said inertia group, then there exists a unique isomorphism of G onto G' with $a_j \rightarrow a'_j$ for $j=1,\cdots,t$. This Remark applies to Theorem 3 of the next section as a special case.

Remark 9. In deducing Theorem 2 above and Theorem 3 of the next section from Theorem 1, one could probably pass to the compositum of $\Omega'_g(V-W)$ and use ramification theory of infinite galois extension.

Remark 11. In this remark, we shall be speaking very roughly and approximately. From the proofs of Theorems 1 and 2, it is clear that if we do not assume V to be simply connected, then the same methods would give us information about the kernel of the natural homomorphism of the fundamental group of $V-W$ onto the fundamental group of V and would imply that this kernel has approximately the description given for the fundamental group of $V-W$ in these theorems. Thus we would get the effect of removing W on the fundamental group of V. In the same vein, if W_1 and W_2 are two subvarieties of V satisfying suitable assumptions, then those methods would give a description for the kernel of the natural homomorphism of the fundamental group of $V-W_1-W_2$ onto the fundamental group of $V-W_1$. We shall exploit these things in a later communication.

Remark 12. The assumption in Theorems 1 and 2 that $\dim |W_j| > 1$ can be replaced by the weaker assumption that $\dim |mW_j| > 1$ for some positive integer m and that there exists a prime divisor in the linear system $|mW_j|$. This remark, in conjunction with our forthcoming work on fundamental groups for branch loci with higher singularities, will be applied to deducing theorems on the nonexistence of irreducible plane curves of a given degree and prescribed singularities.

D. Applications.

13. Theorem of Zariski. As an application of Theorem 1, we shall now deduce the following results which, in the classical case (i. e. for the ground field of complex numbers), is due to Zariski.

THEOREM 3. *Let P_n be the n dimensional projective space over k with $n>1$, let W be a hypersurface in P_n with normal crossings only, let $g^*_1, g^*_2, \cdots, g^*_t$ be the orders of the irreducible components of W. Let $d=1$ in case $p=0$ and $d=$ the highest power of p which divides $g^*_1, g^*_2, \cdots, g^*_t$ in case $p\neq 0$; let $g_j = g^*_j d^{-1}$, and let G be the abelian group on t generators $a_1, a_2, \cdots, a_t$ with the only relation*

$$a_1^{g_1}a_2^{g_2}\cdots a_t^{g_t}=1.$$

Then G is a tame fundamental parent group of $V-W$. Also, $\pi^*(V-W) = \pi'(V-W)$, and hence G is a reduced fundamental parent group of $V-W$ as well. G is a direct product of a free abelian group on $t-1$ generators and a cyclic group of order equal to the greatest common divisor of $g_1, g_2, \cdots, g_t$, i.e., equal to the greatest common divisor of $g^*_1, g^*_2, \cdots, g^*_t$ in case $p=0$ and to the part of this prime to p in case $p\neq 0$.

First, we shall give three lemmas.

LEMMA 32. *Let A be a unique factorization domain with quotient field K such that A contains an alegbraically closed field k of characteristic p. Let Λ be an algebraic closure of K. Let $d_1, d_2, \cdots, d_t$ be pairwise coprime irreducible nonunits in A. Let Z be the set of all positive integers in case $p=0$ and let Z be the set of all positive integers prime to p in case $p\neq 0$. Then we can choose elements $u_m, d_1^{1/m}, d_2^{1/m}, \cdots, d_t^{1/m}$ in Λ such that u_m is a primitive m-th root of 1 and $d_1^{1/m}, d_2^{1/m}, \cdots, d_t^{1/m}$ are m-th roots respectively of $d_1, d_2, \cdots, d_t$ such that if m and m^* are in Z with $m^*\equiv 0 \pmod m$, then $(u_{m^*})^{m^*/m} = u_m$ and $(d_j^{1/m^*})^{m^*/m} = d_j^{1/m}$ for $j=1,2,\cdots,t$. For m in Z, let $L_m = K(d_1^{1/m}, d_2^{1/m}, \cdots, d_t^{1/m})$. Then τ_{mj}: $d_j^{1/m} \rightarrow u_m d_j^{1/m}$, $d_q^{1/m} \rightarrow d_q^{1/m}$ for $q\neq j$ is an automorphism of L_m/K of order m; $G(L_m/K)$ is the direct product of the t cyclic subgroups of order m generated by $\tau_{m1}, \tau_{m2}, \cdots, \tau_{mt}$ respectively, so that $G(L_m/K)$ is an abelian group of order m^t. Let Ω_1 denote the set of all field extension of K contained in L_m for the various m in Z. Then Ω_1 is closed with respect to subfields and finite compositums. Hence the galois groups over K of all the various members of Ω_1 form a group tower π_1. Let F_t be the free abelian group on t generators $\alpha_1, \alpha_2, \cdots, \alpha_t$. Then there exists a unique weak parent map f of F_t onto π_1 such that if for*

M in Ω_1, we denote by $f(M)$ the homomorphism, belonging to f, of F_t onto $G(M/K)$, then we have $f(L_m)(\alpha_j) = \tau_{mj}$ for $j = 1, 2, \cdots, t$ and for all m in Z. Furthermore, if $p = 0$, then f is a parent map and if $p \neq 0$, then f is a modulo p parent map and gives an isomorphism of the modulo p derived group tower of F_t onto π_1.

Proof. The existence of $u_m, d_j^{1/m}$ follows thus: Fix an ascending sequence $1 = m_1 < m_2 < m_3 < \cdots$ of integers in Z such that m_q divides m_{q+1} for all q and each integer in Z divides some m_q. By induction on q, we shall define for $h = 1, 2, \cdots, q$, a primitive m_h-th root u_{m_h} of 1 in Λ and an m_h-th root d_j^{1/m_h} of d_j in Λ such that for all integers g, h with $1 \leqq g \leqq h \leqq q$, we have $(u_{m_h})^{m_h/m_g} = u_{m_g}$ and $(d_j^{1/m_h})^{m_h/m_g} = d_j^{1/m_g}$. For $g = 1$, we set $u_{m_1} = 1$ and $d_j^{1/m_1} = d_j$; now suppose $q > 1$ and assume that this has been done for $q - 1$. Then there exists a primitive m_q-th root v of 1 in Λ and a m_q-th root of d_j in Λ such that $(v)^{m_q/m_{q-1}} = u_{m_{q-1}}$ and $(e_j)^{m_q/m_{q-1}} = d_j^{1/m_{q-1}}$, and we can take v for u_{m_q} and e_j for d_j^{1/m_q}. Now for m in Z dividing a particular m_q, set $u_m = (u_{m_q})^{m_q/m}$ and $d_j^{1/m} = (d_j^{1/m_q})^{m_q/m}$, and observe that this is independent of m_q.

Since d_j is irreducible in A, $K(d_j^{1/m})/K$ is a galois extension with galois group cyclic of order m and generated by $d_j^{1/m} \to u_m d_j^{1/m}$. L_m is the compositum of $K(d_1^{1/m}), \cdots, K(d_t^{1/m})$; hence L_m/K is galois and $G(L_m/K)$ is naturally isomorphic to a subgroup of the direct product of $G(K(d_1^{1/m})/K)$, $\cdots, G(K(d_t^{1/m})/K)$, and hence if we showed that

$$[L_m : K] = m^t = [K(d_1^{1/m}) : K] \cdots [K(d_t^{1/m}) : K],$$

then the assertion about $G(L_m/K)$ would follow. Let v be the valuation of K given by the irreducible element d_1 of the unique factorization domain A with $v(d_1) = 1$ and let v^* be an extension of v to L_m; then $v^*(d_1^{1/m}) = 1/m$, and hence $r(v^* : v) \equiv 0 \pmod{m}$. For $j \neq 1$, d_j and d_1 are coprimes irreducibles, and hence the discriminant of $X^m - d_j$ is of v-value zero so that v is unramified in $K(d_j^{1/m})$. Hence v is unramified in $K(d_2^{1/m}, \cdots, d_t^{1/m})$ which implies that $r(v^* : K(d_2^{1/m}, \cdots, d_t^{1/m})) = r(v^* : v) \geqq m$. Now making induction on m, we can conclude that $[L_m : K] = m^t$.

It is clear that the automorphism τ_{mj} form a consistent family of generators for the cofinal set $\{G(L_m/K), m \in Z\}$ of π_1, and hence, by Lemma 22 of Section 8, we can find a (unique) group G on t generators $a_1, a_2, \cdots, a_t$ with a weak parent map $g = \{g(M) : G \to G(M/K), M \in \Omega_1\}$ such that $g(L_m)(a_j) = \tau_{mj}$ for $j = 1, 2, \cdots, t$ and $m \in Z$. By Proposition 4 of Section 9, G is abelian. Let $n_1, n_2, \cdots, n_t$ be arbitrary integers not all zero and let

$b = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}$. Say $n_1 \neq 0$. Then there exists m in Z with $n_1 \not\equiv 0 \pmod{m}$; let $M = K(d_1^{1/m})$. Since $g(M)$ equals $g(L_m)$ followed by the canonical homomorphism of $G(L_m/K)$ onto $G(M/K)$, we can conclude that $g(M)(a_j) = 1$ for $j = 2, 3, \cdots, t$ and $g(M)(a_1)$ is the automorphism $d_1^{1/m} \to u_m d_1^{1/m}$ and hence is of order m; therefore $g(M)(b) = (a_1^{n_1}) \neq 1$ since $n_1 \not\equiv 0 \pmod{m}$. This shows that G is the free abelian group on $a_1, a_2, \cdots, a_t$ and we can take $G = F_t$, $\alpha_j = a_j$ and $f = g$; the uniqueness follows from Lemma 22 of Section 8.

Now assume that $p \neq 0$. Since every member of Ω_1 is contained in some L_m, it follows that the order of each group in π_1 is prime to p. Observe that in $G(L_m/K)$, $m =$ order of $\tau_{mj} =$ order of τ_{mj} in the quotient group of $G(L_m/K)$ by the subgroup generated by $\tau_{m1}, \tau_{m2}, \cdots, \tau_{mj-1}$; hence Lemma 24 of Section 8 tells us that any normal subgroup of G of finite index prime to p contains $g(L_m)^{-1}(1)$ for some m in Z and hence g is a modulo p quasi parent map of G onto π_1. Now let $n_1, n_2, \cdots, n_t$ be arbitrary integers not all zero and let $b = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}$. Say $n_1 \neq 0$. Then there exists m in Z with $n_1 \not\equiv 0 \pmod{m}$. Let $L = K(d_1^{1/m})$. Then $g(L)(b) = g(L)(a_1)^{n_1} \neq 1$ since the order of $g(L)(a_1)$ is m. Therefore g is a modulo p parent map. If $p = 0$, omitting any reference to p in the above argument, we conclude that g is a parent map.

LEMMA 33. *Let the situation be as Lemma 32. Let $g^*_1, g^*_2, \cdots, g^*_t$ be given positive integers. Let $d = 1$ in case $p = 0$ and let $d =$ the highest power of p which divides $g^*_1, g^*_2, \cdots, g^*_t$ in case $p \neq 0$. Let $g_j = g^*_j d^{-1}$. Let N be the subgroup of F_t generated by*

$$\beta = \alpha_1^{g_1} \alpha_2^{g_2} \cdots \alpha_t^{g_t}.$$

Let q be the canonical homomorphism of F_t onto $G = F_t/N$ and let $a_j = q(\alpha_j)$. Let π' be a subtower of π_1 and let Ω' be the corresponding subset of Ω_1. Assume that $L \in \Omega_1$ and $G(L/K)$ cyclic implies that $L \in \Omega'$ if and only if

$$f(L)(\alpha_1)^{g^*_1} f(L)(\alpha_2)^{g^*_2} \cdots f(L)(\alpha_t)^{g^*_t} = 1. \tag{I}$$

Then

$$N = \bigcap_{L \in \Omega'} f(L)^{-1}(1),$$

so that for each L in Ω', there exists a unique homomorphism $\phi(L)$ of G onto $G(L/K)$ with $f(L) = \phi(L) q$, and $\phi = \{\phi(L), L \in \Omega'\}$ is a weak parent map of G onto π'. Furthermore, if $p = 0$ then ϕ is a parent map, and if $p \neq$ then ϕ is a modulo p parent map. G is a direct product of a free abelian group on $t - 1$ generators and a cyclic group whose order is the greatest common divisor of $g_1, g_2, \cdots, g_t$.

Proof. Let E be a finitely generated abelian group; then E is a direct sum of cyclic subgroups $E_1, E_2, \cdots, E_n$; let e_h be the projection of E onto E_h, then given $x \in E$ with $x \neq 1$, there exists h such that $e_h(x) \neq 1$. Using this argument, we can at once conclude that given *any* $L \in \Omega_1$, $L \in \Omega'$ if and only if (I) holds. Let N^* be the cyclic subgroup of F_t generated by

$$\beta^* = \alpha_1^{g^*_1}\alpha_2^{g^*_2} \cdots, \alpha_t^{g^*_t}.$$

Then for all L in Ω', $N^* \subset f(L)^{-1}(1)$ so that there exists a unique homomorphism $\phi^*(L)$ of $G^* = F_t/N^*$ onto $G(L/K)$ such that $f(L)$ equals the canonical homomorphism of F_t onto G^* followed by $\phi^*(L)$. If $p = 0$, then f is a parent map and hence, invoking Lemma 27 of Section 9, we conclude that $\phi^* = \phi$ is a parent map of $G^* = G$. Now assume that $p \neq 0$. Since f is a modulo p parent map of F_t onto π_1, we conclude that $\phi^* = \{\phi^*(L), L \in \Omega'\}$ is a modulo p quasi parent map of G^* onto π'. Let x be the greatest common divisor of $g^*_1, g^*_2, \cdots, g^*_t$, let p^v be the highest power of p which divides x and let $z = xp^{-v}$. Let $h_j = g^*_j x^{-1}$ and let

$$\gamma = \alpha_1^{h_1}\alpha_2^{h_2} \cdots \alpha_t^{h_t},$$

so that $\beta = \gamma^z$ and $\beta^* = \gamma^x$. Since the greatest common divisor of $h_1, h_2, \cdots, h_t$ is 1, by well known properties of finitely generated free abelian groups, we can find $\gamma_2, \gamma_3, \cdots, \gamma_t$ such that $\gamma = \gamma_1, \gamma_2, \cdots, \gamma_t$ is a free abelian basis of F_t. Let $k = k_1, k_2, \cdots, k_t$ be the images in G^* of $\gamma_1, \gamma_2, \cdots, \gamma_t$ respectively. Then G^* is a direct sum of the free abelian group V_2 generated by the free generates $k_2, k_3, \cdots, k_t$ and the cyclic group V_1 generated by k_1. Let v_j be the projection of G^* onto V_j. For all L in Ω', the order of $\phi^*(L)(k_1)$ is prime to p, and hence it divides z so that $\phi^*(L)(k_1^z) = 1$. Now let $u_1, u_2, \cdots, u_t$ be arbitrary positive integers and let $h = k_1^{u_1}k_2^{u_2} \cdots k_t^{u_t}$. Assume that $\phi^*(L)(h) = 1$ for all L in Ω'. Applying the consideration of the last paragraph in the proof of Lemma 32 to V_2, we conclude, via v_2, that $u_2 = u_3 = \cdots = u_t = 0$. Next, let τ be a homomorphism of the cyclic group V_1 of order x onto a cyclic group of order z. Then τv_1 maps V onto a finite group of order prime to p, and hence there exists L in Ω' such that the kernels of τv_1 and $\phi^*(L)$ coincide. Therefore $\phi^*(L)(k_1)$ is of order z. Since $\phi_0(L)(k_1^{u_1}) = \phi^*(L)(h) = 1$, we conclude that $u_1 \equiv 0 \pmod z$. This shows that $\bigcap_{L \in \Omega'} \phi^*(L)^{-1}(1)$ is generated by k_1^z. Therefore ϕ is a modulo p parent map of G onto π'.

The following lemma is well known; we give it here for the sake of completeness.

LEMMA 34. *The projective n dimensional space P_n over k is simply connected.*

Proof. We shall make induction on n. For $n=1$, this is well known (Proposition 3 of Section 3 of [A3]); now assume that $n>1$ and that P_{n-1} is simply connected. Let K^* be a finite separable algebraic extension of $k(P_n)$ which is unramified over P_n let, $[K^*: k(P_n)]=m$, let V^* be a K^*-normalization of P_n and let ϕ be the rational map of V^* onto P_n. Since the hyperplanes of P_n form a linear system of dimension greater than one and is without fixed components, by the generalized Bertini theorem (Section 4), we can find a hyperplane P_{n-1} in P_n such that $U^*=\phi^{-1}(P_n)$ is irreducible, and since $Q(P_{n-1},P_n)$ is unramified in K^*, we must have $[(U^*): k(P_{n-1})] = [K^*: k(P_n)]=m$ (Section 2). Since k is algebraically closed and K^*/P_n is unramified, for each point of P_n, there are exactly m points on V^*, and hence, for each point of P_{n-1}, there are exactly m points on U^*, and hence $k(U^*)/P_{n-1}$ is unramified. Therefore, by the induction hypothesis, $k(U^*) = k(P_{n-1})$, i.e., $m=1$, i.e., $K^*=k(P_n)$.

Proof of Theorem 3. Since $n>1$, the dimension of the complete linear system determined by an hypersurface in P_n is greater than 1 and any two hypersurfaces in P_n have a point in common; also, by Lemma 34, P_n is simply connected. Therefore, by Theorem 1, each group in $\pi'(P_n-W)$ is abelian and also $\pi^*(P_n-W)=\pi'(P_n-W)$. Now fix an algebraic closure Λ of $K=k(P_n)$. Choose an affine coordinate system $x_1,x_2,\cdots,x_n$ in P_n such that the hyperplane at infinity is not in W. Then $K=k(x_1,x_2,\cdots,x_n)$ and W_j is given by an irreducible polynomial $d_j=d_j(x_1,x_2,\cdots,x_n)$ of degree g^*_j in the polynomial ring $A=k[x_1,x_2,\cdots,x_n]$. Now A is unique factorization domain and $d_1,d_2,\cdots,d_t$ are pairwise coprime irreducible nonunits in A. Hence we can apply Lemma 33; we shall use the notation of that lemma. We shall show that $\Omega'_g(P_n-W)\subset\Omega_1$. Since each member of $\Omega'_g(P_n-W)$ is an abelian extension of K and since an abelian extension is a compositum of cyclic extensions, it is enough to show that each cyclic extension of K contained in $\Omega'_g(P_n-W)$ is contained in Ω_1. So let L be in $\Omega'_g(P_n-W)$ such that L/K is cyclic. Let $[L: K]=m$. Since $\pi'(P_n-W) = \pi^*(P_n-W)$, m is prime to p in case $p\neq 0$, and hence there exists a polynomial

$$X^m-y \tag{1}$$

with y in K such that L is the root field of K in Λ. We can arrange matters so that $y\in A$ and that y is not divisible by the m-th power of any nonunit in A. Then the hyperplanes in P_n given by irreducible factors of y must be

ramified in L, and hence, after multiplying y by a suitable element of k, we have:

$$y = d_1^{v_1} d_2^{v_2} \cdots d_t^{v_t}. \tag{2}$$

Hence $L \subset L_m$, i. e., $L \in \Omega_1$. Thus $\Omega'_g(P_n - W) \subset \Omega_1$.

Next, let L be any member of Ω_1 such that $G(L/K)$ is cyclic. *We assert that $L \in \Omega'_g(P_n - W)$ if and only if*

$$f(L)(\alpha_1)^{g^*_1} f(L)(\alpha_2)^{g^*_2} \cdots f(L)(\alpha_t)^{g^*_t} = 1. \tag{3}$$

Let $[L:K] = m$ and arrange matters so that L is a rootfield of (1), where y is given by (2). Then the part of $\Delta(L/P_n)$ at finite distance is contained in W, and hence $\Delta(L/P_n) \subset W$ if and only if the hyperplane at infinity is not ramified in L. It is easily verified that the hyperplane at infinity is not ramified in L if and only if

$$v_1 g^*_1 + v_2 g^*_2 + \cdots + v_t g^*_t \equiv 0 \pmod{m}. \tag{4}$$

Note that $L \subset L_m$ and that $f(L_m)(\alpha_j) = \tau_{mj}$; let e be the canonical homomorphism of $G(L_m/K)$ onto $G(L/K)$. Then $f(L) = ef(L_m)$, and hence (3) is equivalent to

$$\prod_{h=1}^{t} e(\tau_{mh})^{g^*_h} = 1, \tag{3'}$$

i. e., to

$$e[\prod_{h=1}^{t} \tau_{mh}{}^{g^*_t}] = 1, \tag{3*}$$

i. e., to

$$\tau = \prod_{h=1}^{t} \tau_{mh}{}^{g^*_h} \in e^{-1}(1) = G(L_m/L) \subset G(L_m/K). \tag{5}$$

Thus we have to show that (4) is equivalent to (5). Next,

$$d = \prod_{h=1}^{t} (d_h{}^{1/n})^{v_h}$$

is a primitive element of L/K, and hence the automorphism τ of L_m/K is in $G(L_m/L)$ if and only if $\tau(d) = d$. Now

$$\begin{aligned}
\tau(d) &= \prod_{j=1}^{t} \tau_{mj}{}^{g^*_j} [\prod_{h=1}^{t} (d_h{}^{1/m})^{v_j}] \\
&= \prod_{j=1}^{t} [\tau_{mj}{}^{g^*_j} (\prod_{h=1}^{t} (d_h{}^{1/m})]^{v_j} \\
&= \prod_{j=1}^{t} [u_m{}^{g^*_j} (d_j{}^{1/m})]^{v_j} \\
&= \prod_{j=1}^{t} [u_m{}^{v_j g^*_j} (d_j{}^{1/m})^{v_j}] \\
&= [(u_m)^{v_1 g^*_1 + v_2 g^*_2 + \cdots + v_t g^*_t}] d.
\end{aligned}$$

Hence $\tau(d) = d$ if and only if

$$(u_m)^{v_1 g^*_1 + v_2 g^*_2 + \cdots + v_t g^*_t} = 1,$$

i.e., if and only if (4) holds. This proves the italicized assertion. Now the theorem follows from Lemma 33.

Remark 13. Referring to Theorem 3, let G^* be the abelian group generated by t generators $a^*_1, a^*_2, \cdots, a^*_t$ and the only relation

$$a^{*\,g^*_1}_1 a^{*\,g^*_2}_2 \cdots a^{*\,g^*_t}_t = 1.$$

Then it follows from the proof of Lemma 33 that for $p = 0$, $G^* = G$, i.e., G^* is a (tame, reduced) fundamental parent group of $P_n - W$; and for $p \neq 0$, G^* is a modulo p quasi parent group of $\pi'(P_n - W) = \pi^*(P_n - W)$.

14. Theorem of Picard. As another application of our main results, we shall deduce a result (Theorem 5 below) which, for dimension two in the classical case, was asserted by Picard. In the following Definition, and in Lemmas 35, 36 and Theorem 4, K is an n dimensional algebraic function field over k, Λ is a fixed algebraic closure of K and V is a normal projective model of K/k.

Definition. Let W be an irreducible $n-1$ dimensional subvariety of V. Since linear equivalence preserves degress (in the embedding projective space of V), it follows that the positive integers m such that there exists a divisor D on V with $W \equiv mD$ are bounded, the maximum of these integers will be called *the embedding degree of W in V and will be denoted by $\delta(W, V)$.* Also, we define *the reduced embedding degree $\delta(W, V)$ of W in V* by setting it equal to $\delta^*(W, V)$ in case $p = 0$ and equal to $\delta^*(W, V)$ divided by the highest power of p which divides $\delta^*(W, V)$ in case $p \neq 0$. It is obvious from the definition that $\delta^*(W, V)$ and hence $\delta(W, V)$ is a biregular invariant. The biregular invariance of $\delta(W, V)$ will also follow from Lemma 36 and from that lemma, it will also follow that $\delta(W, V)$ equals the maximum as well as the least common multiple of all integers m prime to p (respectively, all integers m) in case $p \neq 0$ (respectively, $p = 0$) for which there exists a divisor D on V with $W \equiv mD$. Also note that if V is a projective space P_n, then $\delta^*(W, V)$ is the usual order of the hypersurface of W; and in this case, $\delta(W, V)$ will also be called *the reduced order of W.*

LEMMA 35. *Let V^* be a normalization of V in a finite separable algebraic extension K^* of K and assume that V^* is simply connected. Then*

any one of the following two conditions implies that V is simply connected: (1) there exist an irrreducible subvariety of V^* whose ramification index over K equals $[K^*: K]$; (2) any field between K and K^* other than K is ramified over V.

Proof. In view of Lemma 4 of Section 2 of [A2], (1) implies (2). Now assume (2). Let K_1 be any finite separable algebraic extension of K such that K_1/V is unramified. Let K^*_1 be a compositum of K^* and K_1. Then by Lemma 9 of Section 2, K^*_1/V^* is unramified. Hence $K^*_1 = K^*$, i.e., $K_1 \subset K^*$ which, in view of (2), implies that $K_1 = K$.

LEMMA 36. *Let W be an irreducible $n-1$ dimensional subvariety of V. Assume that V is nonsingular and simply connected. Let K^* be the compositum of all finite abelian extensions of K which are tamely ramified over V and for which the branch locus over V is contained in W. Then K^*/K is a cyclic extension of degree $\delta(W, V)$.*

Proof. Let L/K be a finite abelian extension in Λ such that L/V is tamely ramified and $\Delta(L/V) \subset W$; let V^* be a L-normalization of V and let ϕ be the rational map of V^* onto V. Since L/K is abelian and since the inertia groups over K of the various irreducible components of $\phi^{-1}(W)$ are K-conjugates, all these inertia groups must be the same; let L^* be the fixed field of this inertia group. There L/L^* is cyclic (Lemma 14, Section 2), and L^*/V is unramified (Lemmas 13 and 17, Section 2); since V is simply connected, we have $L^* = K$; *this also shows that $[L: K]$ = ramification index over K of any irreducible component of $\phi^{-1}(W)$.*

Therefore it is enough to show that for an integer $m > 1$, which is prime to p in case $p \neq 0$, there exists a cyclic extension L/K of degree m such that L/V is tamely ramified and $\Delta(L/V) \subset W$ if and only if there exists a divisor D and V with $W \equiv mD$. Assume that L exists; then L/K is the root field of a polynomial $X^m - y$ with $y \in K$. Then $\Delta(L/V)$ contains each of the prime divisors which occur in the divisor (y) of the function y with a coefficient which is not divisible by m. By the above italicized remark, it follows that W occurs in (y) with a coefficient q which is prime to m. Then we can find an integer q^* prime to m such that $qq^* \equiv 1 \pmod{m}$. Replacing y by y^q, we can assume that $q \equiv 1 \pmod{m}$, which implies that $(y) - W$ is equal to m times a divisor D so that $W \equiv mD$. Conversely, assume that D exists. Then there exists y in K such that $(y) = W - mD$, and we may take L to be the root field over K of the polynomial $X^m - y$.

THEOREM 4. *Let W be an irreducible subvariety of V. Assume that W*

has only normal crossings and dim $| W | > 1$ *and that* V *is simply connected. Let* K^* *be the compositum of all the fields in* $\Omega'_g(V - W)$. *Then* (i) K^*/K *is cyclic of degree* $\delta(W, V)$; *and* $V - W$ *has as a tame (as well as reduced) fundamental parent group a cyclic group of order* $\delta(W, V)$. *Let* V^* *be a* K^**-normalization of* V *and let* ϕ *be the map of* V^* *onto* V (*so that* $V^* - \phi^{-1}(W)$ *is the "tame universal covering" of* $V - W$). *Then* (ii) $V^* - \phi^{-1}(W)$ *is tamely simply connected. Finally,* (iii) *the normalization of* V *in any field between* K *and* K^* (*in particular,* V^*) *is simply connected.*

Proof. That each group in $\pi'(V - W)$ is abelian is exactly Theorem 1 for $t = 1$. However, the argument for $t = 1$ is rather easier and is briefly thus: For K_1 in $\Omega'_g(V - W)$, let V_1 be a K_1-normalization of V and let ϕ_1 be the rational map of V_1 onto V. Then by Proposition 6 of Section 11, $W_1 = \phi_1^{-1}(W)$ is irreducible, and hence K is itself the splitting field of W_1/W. Then K_1/K_2 is cyclic (Lemma 14, Section 2) and K_2/V is unramified (Lemmas 13 and 17, Section 2); since V is simply connected, we have $K_2 = K$. Now (i) follows from Lemma 36 above. (ii) follows from (i) in view of Lemma 9 of Section 2. From (i), it follows that V^* is simply connected (Lemma 29 of Section 10) and this, together with Lemma 35 above and the italicized statement in the proof of Lemma 36, gives (iii).

PROPOSITION 7. *Let* V^* *be a normal projective variety over* k. *Assume that there exists a rational map* ϕ *of* V^* *onto a nonsingular projective simply connected variety* V *of finite index such that:* (1) ϕ *and* ϕ^{-1} *are both free from fundamental points,* (2) V^*/V *is tamely ramified,* (3) $\Delta(V^*/V)$ *is irreducible,* (4) $\Delta(V^*/V)$ *has only normal crossings, and* (5) dim $|\Delta(V^*/V)| > 1$. *Then* V^* *is simply connected,* $k(V^*)/k(V)$ *is galois with galois group cyclic of order dividing* $\delta(\Delta(V^*/V), V)$, *and* $V^* - \phi^{-1}(\Delta(V^*/V))$ *is tamely simply connected in case* $[k(V^*) : k(V)] = \delta(\Delta(V^*/V), V)$.

Proof. This is essentially Theorem 4 stated from a covering to the projection instead of the other way around.

PROPOSITION 8. *Let* W *be an irreducible hypersurface of reduced degree* g *with normal crossings only in projective* n *dimensional space* P_n *over* k. *Let* K^* *be the compositum of all the fields in* $\Omega'_g(P_n - W)$. *Then* (i) $K^*/k(P_n)$ *is cyclic of degree* g *so that* $P_n - W$ *has for a tame (as well as reduced) fundamental parent group a cyclic group of order* g. *Let* V^* *be a* K^**-normalization of* P_n *and let* ϕ *be the rational map of* V^* *onto* P_n. *Then* (ii) $V^* - \phi^{-1}(W)$ *is tamely simply connected. Finally,* (iii) *the normaliza-*

tion of P_n in any field between $k(P_n)$ and K^* (in particular, V^*) is simply connected.

Proof. For $n = 1$, this means that $P_1 - W$ (W is a point) is tamely simply connected and P_1 is simply connected; this is well known (Proposition 6 of Section 3 of [A3]); now assume that $n > 1$. Then this proposition follows from Theorem 4 in view of Lemma 34 of Section 13 or, alternatively, (i) is exactly Theorem 3 for $t = 1$ and (i) and (iii) follow from (i) as in the proof of Theorem 4.

THEOREM 5. *Let V^* be a hypersurface in projective $n+1$ dimensional space P_{n+1} having an affine equation*

$$X_{n+1}{}^m - f(X_1, X_2, \cdots, X_n) = 0,$$

where $W\colon f(X_1, X_2, \cdots, X_n) = 0$ is an irreducible hypersurface (i.e., f is an irreducible polynomial) in projective n space P_n (with affine coordinates $X_1, X_2, \cdots, X_n$) having only normal crossings (in particular, say, W is to be nonsignular, or 'generic'), such that m divides the reduced order g of W. Then V^* is simply connected. If $m = g$, then $V^* - (f(X_1, X_2, \cdots, X_n) = 0 \cap V^*)$ is tamely simply connected.

Proof. Project V^* on P_n by the natural projection and call this projection map ϕ. By the Jacobian criterion, the singularities of V^* lie above the singularities of W, and hence the singular locus of X^* is of dimension less than $n - 1$. Therefore V^* is normal. That ϕ and ϕ^{-1} are free from fundamental points is obvious. Now either apply Proposition 8 or apply Proposition 7 together with Lemma 34 of Section 13.

E. Classical Case and Motivation.

In this chapter, C will denote the field of complex numbers, no reference will be made to the Zariski topology, V will denote a normal projective model of an n dimensional algebraic function field over an algebraically closed ground field k of characteristic p and W will denote a proper subvariety of V. Also, π_1 will denote the usual topological fundamental group of a topological space, and furthermore, $\Gamma\pi_1$ will denote the intersection of all subgroups of π_1 of finite index and $\gamma\pi_1$ will denote the factor group $\pi_2/\Gamma\pi_1$. In this chapter, since its main purpose is motivation and description, we shall not try to be completely precise.

15. Existence of algebraic coverings. Suppose $k=C$, so that V becomes a topological space in the classical manner. Then $V-W$ is connected, and finite regular unramified topological coverings of $V-W$ are in one to one natural correspondence with finite homomorphic images of $\pi_1(V-W)$. Let V' be a finite unramified topological covering of $V-W$. Then by the recent work of Grauert and Remmert ([GR], see also Enriques' work on this topic in [E] and Chapter VIII of [Z2])—which can be called the general Riemann Existence Theorem—implies that V' can be uniquely completed to a normal algebraic variety V^*, thus making V^* an algebraic covering of V with $\Delta(V^*/V) \subset W$. Now if the covering V' is regular, then $k(V^*)/k(V)$ is galois and the covering group of V' over V is naturally isomorphic to the galois group $G(k(V^*)/k(V))$, and hence we can conclude that $\gamma\pi_1(V-W)$ is, in a natural way, a parent group of $\pi(V-W) = \pi'(V-W) = \pi^*(V-W)$. This explains our terms, . . . fundamental group tower . . ., in abstract algebraic geometry. Even if the Existence Theorem were not available, these group towers would, in the abstract case carry the same weight; however, in the presence of the Existence Theorem, these concepts do carry more weight for considerations of the classical case, and, for instance, from Theorem 2 (Section 12), we can at once conclude the following new result for the classical case.

PROPOSITION 9. *Suppose $k=C$. Assume that V is nonsingular, W is pure $n-1$ dimensional with irreducible components $W_1, W_2, \cdots, W_t$, and that:* (1) $\dim |W_j| > 1$ *for* $j=1,2,\cdots,t$; (2) *W has only normal crossings;* and (3) *V has no finite unramified topological coverings (this would certainly be so if V is topologically simply connected, i.e., if $\pi_1(V)=1$). Then $\gamma\pi_1(V-W)$ is t-step nilpotent and it is abelian in case $W_1, W_2, \cdots, W_t$ are pairwise connected (here topological connectedness and algebrogeometric connectedness are the same). Also $V-W$ has an unrestricted tame fundamental parent group, namely $\gamma\pi_1(V-W)$.*

Now consider Theorem 3 of Section 13. In the classical case, this was proved by Zariski, namely in [Z1], also Chapter VIII of [Z2]; he proved the following:

(A) *Let W be a curve with only normal crossings in the complex projective plane P_2 and let $g_1, g_2, \cdots, g_t$ be the orders of the irreducible components of W. Then $\pi_1(P_2-W)$ is an abelian group with t generators $a_1, a_2, \cdots, a_t$ and the only relation*

$$a_1^{g_1}a_2^{g_2}\cdots a_t^{g_t}=1.$$

Then in [Z3], he proved the following theorem.

(B) *Let W be a hypersurface in complex projective n space P_n and let P_{n-1} be a generic hypersurface in P_n. Then*

$$\pi_1(P_n - W) \text{ and } \pi_1(P_{n-1} - P_{n-1} \cap W)$$

are naturally isomorphic.

Putting together (A) and (B), invoking the Existence Theorem of Grauert-Remmert, and observing that in a finitely generated abelian group, the intersection of subgroups of finite index is 1 (Lemma 27 of Section 9), there results Theorem 3 in case $k = C$. Also note that Theorem 3 gives an evidence of (B) in the abstract case.

Next, consider Theorem 5 of Section 14. For $n = 2$ and $k = C$, this was asserted by Picard (Sections 12-14 of Chapter IV of [PS]) in connection with his statement that any nonsingular surface in complex projective three space is simply connected. Note that in the classical case, in view of (B) above, the theorem for general n follows from the case $n = 2$.

16. Finite generation of fundamental groups. Consider the following statement which one expects to be true: (α) *Let W be a subvariety of an algebraic variety V over C; then V can be triangulated so as to make W a subcomplex.* Now (α) implies that $\pi_1(V - W)$ is finitely generated; for instance, take the third barycentric subdivision of V, then $V - W$ can be 'projected' onto a (closed) subcomplex X of $V - W$, thus making X a deformation retract of $V - W$; hence $\pi_1(V - W) = \pi_1(X)$ and, X being a finite complex, $\pi_1(X)$ is the fundamental group of a 'tree' and hence is finitely generated. Hence the group tower $\pi(V - W) = \pi'(V - W) = \pi^*(V - W)$ has a finitely generated parent group and, in particular, a finitely generated weak parent group. This is the reason why we have included *finite generation* in the definitions of (1) a tame fundamental parent group, (2) a reduced fundamental parent group, (1*) a tame fundamental weak parent group, and (2*) a reduced fundamental weak parent group. The reason for not at all defining, in case of $p \neq 0$, a fundamental parent (respectively: weak parent) group, say as a finitely generated unrestricted fundamental parent (respectively: weak parent) group, is that the entire group tower (including tame as well as untame coverings) can be way too large(as is exhibited in [A3, 4], and in general we do not expect it to have a finitely generated parent or weak parent group.

The explanation of the concepts (1*) and (2*) is now complete. However, concerning the concepts of (1) a tame fundamental parent group G of $V-W$ and (2) a reduced fundamental parent group G^* of $V-W$, we must explain one point, namely that in case of nonzero characteristic $p\neq 0$, G (respectively: G^*) is required to be just a little less than a finitely generated parent group of $\pi'(V-W)$ (respectively: $\pi^*(V-W)$), namely we have required that G (respectively: G^*) be finitely generated and that there exist a weak parent map f of G (respectively: G^*) onto $\pi'(V-W)$ (respectively: $\pi^*(V-W)$) such that the kernel of f include all the normal subgroups of G (respectively: G^*) of finite index *prime to* p. The reason for this is that in general there does not exist a finitely generated parent group of $\pi'(V-W)$ (respectively: $\pi^*(V-W)$); for instance, certainly we can have a situation in which $\pi'(V-W)$ and $\pi^*(V-W)$ are isomorphic to the modulo p derived group tower of an infinite cyclic group—for instance, take V to be the projective line and for W, take two points (Proposition 6 of Section 3 of [A3]), or take V to be the projective plane and for W, take two lines (Theorem 3 of Section 13)—and this group does not have any finitely generated parent group (Example 2 of Section 7).

Thus the complete analogue for the abstract case of the existence in the classical case of a topological fundamental group which one expects to be always finitely generated is the following statement which we state as a conjecture.

CONJECTURE 1. *For any normal projective algebraic variety V and a subvariety W, there exists a tame fundamental parent group of $V-W$.*

A proof of this conjecture would be a good contribution to the theory of coverings in the abstract case. Note that the existence of a tame fundamental parent group implies the existence of a reduced fundamental parent group. Now if one proves Conjecture 1, then out of the class of all tame fundamental parent groups, how far one can choose one (or more) which one would like to call *a fundamental group* of $V-W$, is another matter—for instance, can one somehow tell the *right* number of generators—, perhaps in the abelian case this is rather easy. A somewhat weaker form of Conjecture 1 is this:

CONJECTURE 2. *For any normal projective algebraic variety V and a subvariety W, there exists a tame fundamental weak parent group of $V-W$.*

Note that for $V-W$, an unrestricted tame (respectively: reduced) fundamental weak parent group exists trivially, for instance, the inverse limit

of $\pi'(V-W)$ (respectively: $\pi^*(V-W)$) will do. In general, the possible existence of an unrestricted tame (respectively: reduced) fundamental parent group does not follow from such general considerations. However, note that in the situation of Theorem 2 of Section 12, in case the W_j are pairwise connected (and hence, in particular, in the situation of Theorem 3 of Section 13), $\pi'(V-W)=\pi^*(V-W)$ has a weak parent group which is finitely generated and abelian; now it is easily shown that if a compact abelian group G has a finitely generated dense subgroup, then every subgroup of G of finite index is closed; hence the inverse limit of $\pi'(V-W)=\pi^*(V-W)$ is an unrestricted tame (as well as reduced) fundamental parent group of $V-W$.

Also recall the result given in Remark 9 of Section 6 of [A3] (see also footnote 4 of the present paper): For any normal projective variety V and a subvariety W, there exist only a finite number of tame coverings of V of a given degree with branch loci contained in W. Note that this is a trivial consequence of Conjecture 2.

Now in this paper, we have given affirmative answers to these conjectures in certain fairly general situations, namely, in the situation of Theorem 2 (Sceion 12), we have affirmed Conjecture 2, and in Remark 8 (Section 12), we have affirmed Conjecture 3 and in the situation of Theorem 3 (Section 13), we have affirmed Conjecture 1. Returning to statement (α): the 'proof' of this given by van der Waerden [V, Appendix to Chapter III] has recently been pointed out by Whitney [W, Footnote 1] to be not entirely correct (for nonsingular V and empty W, (α) is well known). Hence, at least at the present time, the following is another contribution to the classical case.

PROPOSITION 10. *In the situation of Projosition* T (*Section* 15), *the derived group tower of* $\pi_1(V-W)$ *has a finitely generated* (*by t generators*) *parent group, or equivalently, the inverse limit of* $\gamma\pi_1(V-W)$ *contains a finitely generated dense subgroup.*

17. Miscellaneous remarks.

Remark 14. It is clear that the conjectures made in Section 4 of [A3] concerning a comparison between ramification theory in nonzero characteristic p and the ramification theory in the *corresponding* zero characteristic situation can onw be refined using the concepts of this paper; for instance, Conjecture 2 of [A3] would now be refined to read: Let S_p be a situation in nonzero characteristic p and let S_0 be the corresponding situation in characteristic zero. Then:

(The subtower of the fundamental group tower of S_0 consisting of all the members whose order is prime to p)

$$= \text{(the reduced fundamental group tower of } S_p)$$
$$\subset \text{(the tame fundamental group tower of } S_p)$$
$$\subset \text{(the fundamental group tower of } S_0),$$

where inclusion stands for being a subtower.

It is obvious that the results of the present paper give some evidence in support of this refined conjecture; to give one instance, this conjecture has been verified in the situation of Theorem 3 of Section 13; see Remark 13 of Section 13.

Remark 15. Suppose $k = C$. We do not know if the intersection of subgroups of finite index of $\pi_1(V - W)$ is 1. Observe that this is not a consequence of the (almost certain) statement that these groups are finitely generated, or even finitely presented (Chapter VIII of [Z2], page 56 and Appendix G of [K2], and [H]). Note that every finitely presented group can be realized as the fundamental group of a four dimensional real manifold (Example 3 on page 180 of [ST]). However, one does not know which groups can be the fundamental groups of algebraic varieties (complete or not).

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RATIONAL POINTS OF ABELIAN VARIETIES OVER FUNCTION FIELDS.*

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We intend to give here a systematic and simplified exposition of the theorem of the base for divisors [5]. Since the appearance of [5], the theory of the Picard and Albanese varieties has become available, as well as Chow's theory of algebraic systems of abelian varieties (the K/k-trace and image). Using first an elementary equivalence criterion [9], we reduce the theorem of the base on a variety (projective, non-singular in codimension 1) to a theorem concerning the group of rational points of an abelian variety defined over a function field, by means of the above mentioned theories. We then prove the finiteness statement in two steps as usual: First, the so-called weak Mordell-Weil theorem, for which we reproduce the proof given in [3], and second the infinite descent.

We also take this opportunity of reproducing simultaneously a proof of the ordinary Mordell-Weil theorem concerning the group of rational points of an abelian variety defined over a number field, to the effect that this group is finitely generated. This proof is contained in § 3, § 5, §6, and § 7.

Aside from the above mentioned equivalence criterion, which is used only in § 1, we assume only that the reader is familiar with the basic theory of abelian varieties, as it is done for instance in [2]. We do not need the full theory of distributions, and reproduce the definition of the height of a point, together with its elementary functorial properties. This is all that is needed to carry out the infinite descent. We thus make this paper independent of [5], [6], [8].

1. The theorem of the base. If G denotes a set of geometric objects, we shall denote by G_k the subset of these objects which are rational over a field k.

Let V be a projective variety, non-singular in codimension 1, and defined over an algebraically closed field k which we may take to be a

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universal domain. The group of divisors $D(V)_k$ contains the usual subgroups of divisors which are algebraically equivalent to 0 and linearly equivalent to 0 respectively.

$$D(V)_k \supset D_a(V)_k \supset D_l(V)_k.$$

The Picard group $D_a(V)_k/D_l(V)_k$ can be given the structure of an abelian variety, the Picard variety. The theorem of the base asserts that *the factor group $D(V)_k$ modulo $D_a(V)_k$ is of finite type, i.e. finitely generated.* We are going to show here how this theorem can be reduced to a theorem concerning abelian varieties.

Let C_u be a generic curve on V over k, depending on generic linear parameters u. This means that there is a generic linear variety L_u over k such that $C_u = V \cdot L_u$. (Cf. [1], Ch. 7.) Let J be its Jacobian, defined over the field $k(u) = K$. Let $D_0(V)_k$ be the subgroup of $D(V)_k$ consisting of those divisors $X \in D(V)_k$ such that $X \cdot C_u$ is of degree 0. Then $D(V)_k/D_0(V)_k$ is infinite cyclic, and it will suffice to prove that $D_0(V)_k/D_a(V)_k$ is of finite type.

Let $\phi: C_u \to J$ be a canonical map of C_u into its Jacobian, defined over the algebraic closure of $k(u)$. Then

$$\mathfrak{a} \to S(\phi(\mathfrak{a}))$$

is the canonical map of $D_a(C_u) = D_0(C_u)$ into J. We have a homomorphism

$$h: D_0(V)_k \to J_{k(u)}$$

given by the formula

$$h(X) = S(\phi(X \cdot C_u)).$$

According to the elementary equivalence criterion, one knows that the kernel E of h, which contains $D_l(V)_k$, is of finite type modulo $D_l(V)_k$. Hence the factor group $[E + D_a(V)_k]/D_a(V)_k$ is of finite type. To prove the theorem of the base, it will therefore suffice to prove that $D_0(V)_k/[E + D_a(V)_k]$ is of finite type.

The inverse image $h^{-1}(h(D_a(V)_k)$ is precisely equal to $E + D_a(V)_k$. We have therefore an injection

$$0 \to D_0(V)_k/[E + D_a(V)_k] \to J_{k(u)}/hD_a(V)_k.$$

Now let $\psi: V \to A$ be a canonical map of V into its Albanese variety. For a suitable constant $b \in A$, we have a commutative diagram

$$\begin{tikzcd} C_u \arrow[r, "i"] \arrow[d, "\phi"'] & V \arrow[d, "\psi"] \\ J \arrow[r, "i_* + b"'] & A \end{tikzcd}$$

where i_* is the induced homomorphism.

Let us look at the inverse images of divisors in $D_a(A)_k$ under the composite maps $(i_* + b) \circ \phi$ and $\psi \circ \mathrm{i}$. The formalism $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ can be applied to the first according to Appendix 1 of [2] since J and A are non-singular. A direct verification using intersection theory (associativity and the definition of $C_u = V \cdot L_u$) shows that it can also be applied to the second. According to the definition of the Picard variety, which one knows is obtained by pull back from the Albanese variety, we see that the map

$$h: D_a(V)_k \to J_{k(u)}$$

induces the rational homomorphism ${}^t i_*$ on $\hat{A}_k = D_a(V)_k / D_l(V)_k$, if we denote as usual by the upper index t the transpose of i_* on the Picard varieties. Consequently we have an injection

$$0 \to D_0(V)_k / [E + D_a(V)_k] \to J_{k(u)} / {}^t i_* \hat{A}_k.$$

By Chow's theory of the K/k-trace (whose definition is recalled below), one knows that $(\hat{A}, {}^t i_*)$ is a $k(u)/k$-trace of $\hat{J} = J$ ([2], Ch. 8, Th. 12). Consequently, to prove the theorem of the base, it will suffice to prove the following result.

THEOREM 1. *Let K be a finitely generated regular extension of a field k. Let A be an abelian variety defined over K, and let (B, τ) be its K/k-trace. Then $A_K / \tau B_k$ is of finite type.*

For the convenience of the reader, we recall that a couple (B, τ) consisting of an abelian variety B defined over k and an injective (i.e. purely inseparable) homomorphism $\tau: B \to A$ defined over K is said to be a *K/k-trace* of A if it satisfies the following universal mapping property: For any abelian variety C defined over an extension E of k which is free from K over k, and a homomorphism $\alpha: C \to A$ defined over KE, there exists a homomorphism $\alpha': C \to B$ defined over E such that the following diagram is commutative.

$$\begin{tikzcd} C \arrow[rr, "\alpha"] \arrow[dr, "\alpha'"'] & & A \\ & B \arrow[ur, "\tau"'] & \end{tikzcd}$$

Essentially the same arguments which will be used to prove Theorem 1 will also give us

THEOREM 2. *Let K be an algebraic number field, of finite degree over the rationals. Let A be an abelian variety defined over K. Then A_K is finitely generated.*

From these two theorems we recover immediately the fact that if K is finitely generated over the prime field, then A_K is also finitely generated. It suffices to apply the two theorems to K and the algebraic closure of the prime field in K, viewed as a constant field.

2. Reduction steps. The propositions which we prove in this section will be used to reduce our main theorem (Theorem 1) to special cases which are technically easier to handle. Throughout this section, K will denote a finitely generated regular extension of a field k. By a regular extension, we shall always mean a finitely generated one.

We first show that to prove our main theorem, we may extend our function field by a finite separable extension.

PROPOSITION 1. *Let $L \supset K$ be a finite separable extension of K, also regular over k. Let A be an abelian variety defined over K. Let $(A^{K/k}, \tau_K)$ be its K/k-trace and $(A^{L/k}, \tau_L)$ its L/k-trace. Then the factor group*

$$[\tau_L(A^{L/k})_k \cap A_K]/\tau_K(A^{K/k})_k$$

is finite.

Proof. Let Γ_L be the graph of τ_L. It is defined over L. Let us take the intersection $H = \bigcap_\sigma \Gamma_L{}^\sigma$ of Γ_L and its conjugates over K, where σ ranges over the distinct isomorphisms of L over k. Then H is K-closed and is an algebraic subgroup of $A^{L/k} \times A$. Its connected component H_0 is therefore defined over K by Chow's theorem ([2], Ch. II).

$$\Gamma_L \cap [(A^{L/k})_k \times A_K] = H \cap [(A^{L/k})_k \times A_K]$$

and this group contains $H_0 \cap [(A^{L/k})_k \times A_K]$ as a subgroup of finite index, since H_0 is of finite index in H.

We have a surjective homomorphism

$$\Gamma_L \cap [(A^{L/k})_k \times A_K] \to \tau_L(A^{L/k})_k \cap A_K$$

by projection on the second factor. We contend that the inverse image of $\tau_K(A^{K/k})_k$ contains $H_0 \cap [(A^{L/k})_k \times A_K]$.

$$H_0 \cap [(A^{L/k})_k \times A_K] \to \tau_K(A^{K/k})_k$$

From this it will be clear that the desired factor group is finite.

Since H_0 is contained in the graph of a homomorphism of $A^{L/k}$ into A, it is itself the graph of a homomorphism β of its projection B on the first factor into A. Furthermore, B is contained in $A^{L/k}$, is defined over K, and hence over k by Chow's theorem. By the universal property of the K/k-trace, there exists a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "\beta"] \arrow[dr, "\beta'"'] & & A \\ & A^{K/k} \arrow[ur, "\tau_K"'] & \end{tikzcd}$$

with β' defined over k, and hence $\beta(B_k)$ is contained in $\tau_K(A^{K/k})_k$. This proves our contention, and concludes the proof of our proposition.

COROLLARY. *If Theorem 1 is true for L/k, then it is true for K/k, i.e. if $A_L/\tau_L(A^{L/k})_k$ is finitely generated, so is $A_K/\tau_K(A^{K/k})_k$.*

Proof. This is immediate from the proposition and the fact that we have an injection

$$0 \to A_K/[\tau_L(A^{L/k})_k \cap A_K] \to A_L/\tau_L(A^{L/k})_k.$$

Next, we show that we may extend the constant field.

PROPOSITION 2. *Let k^* be any extension of k which is independent of K, and let $K^* = Kk^*$. Let A be an abelian variety defined over K and let (B,τ) be its K/k-trace. Then*

$$A_K \cap \tau B_{k^*} = \tau B_k.$$

Proof. From the definition of the trace, it is clear that (B,τ) is also a K^*/k^*-trace of A. Our assertion is obvious if k^* is separable over k. Hence it suffices to deal with a finite purely inseparable extension k^* of k. The inclusion $\tau B_k \subset A_K \cap \tau B_{k^*}$ is obvious. Conversely, let b be a point of B_{k^*} such that τb is rational over K. If we knew that τ is regular, i.e. that it is birational biholomorphic between B and its image in A, then in view of the fact that τ is defined over K, we would see immediately that $\tau^{-1}\tau b$ is rational over k. As we do not know that τ is regular, we use Chow's regularity theorem, according to the standard technique of [2], Ch. 9. We can write $K = k(u)$ for suitable parameters u, and $A = A_u$, $\tau = \tau_u$. Let $u_1, \cdots, u_m$ be independent generic specializations of u over k. For m large, the map

$$x \to (\tau_{u_1}x, \cdots, \tau_{u_m}x)$$

of B into $A_{u_1} \times A_{u_2} \times \cdots \times A_{u_m}$ is regular, and thus biholomorphic between

B and its image. If τb is rational over K, then $(\tau_{u_1}b, \cdots, \tau_{u_m}b)$ is rational over $k(u_1, \cdots, u_m)$, hence b is rational over $k(u_1, \cdots, u_m)$, hence b is rational over that field. Since b is purely inseparable over k, and since $k(u_1, \cdots, u_m)$ is regular over k, it follows that b is rational over k. This concludes the proof.

COROLLARY. *If Theorem* 1 *is true for K^*/k^*, then it is true for K/k.*

Proof. The factor group $A_K/\tau B_k$ can be identified with a subgroup of $A_{K^*}/\tau B_{k^*}$.

Finally, it will be convenient to deal with the case where K is a function field of dimension 1 over k, i. e. the function field of a curve, and the next proposition shows that the reduction to this case is trivial.

PROPOSITION 3. *Let $K \supset E \supset k$ be a tower of fields such that K is regular over E and E regular over k. If Theorem* 1 *is true for K/E and E/k, then it is true for K/k.*

Proof. Let A be as in Theorem 1, an abelian variety defined over K. Let $(A^{K/E}, \tau_{K/E})$ be a K/E-trace of A, and $(A^{E/k}, \tau_{E/k})$ a E/k-trace of $A^{K/E}$. By the universal mapping property, there is a purely inseparable homomorphism $\beta\colon A^{E/k} \to A^{K/k}$ defined over k such that the following diagram is commutative:

$$\begin{tikzcd}
A & \\
A^{K/E} \arrow[u, "\tau_{K/E}"] & A^{K/k} \arrow[ul, "\tau_{K/k}"'] \\
A^{E/k} \arrow[u, "\tau_{E/k}"] \arrow[ur, "\beta"] &
\end{tikzcd}$$

All the homomorphisms τ are purely inseparable, and thus injective. We have

$$A_K \supset \tau_{K/E}(A^{K/E})_E \supset \tau_{K/E}\tau_{E/k}(A^{E/k})_k = \tau_{K/k}\beta(A^{E/k})_k.$$

If we assume that $(A^{K/E})_E$ is of finite type modulo $\tau_{E/k}(A^{E/k})_k$, it follows that A_K modulo $\tau_{K/k}\beta(A^{E/k})_k$ is also of finite type. But this last group is contained in $\tau_{K/k}(A^{K/k})_k$. We see therefore that it suffices to prove Theorem 1 for each step K/E and E/k of our tower.

It is well known and easy to show that one can always construct a tower $E_0 = k \subset E_1 \subset \cdots \subset E_n = K$ such that each step E_i/E_{i-1} is regular and of transcendence degree 1. This comes from the lemma of Bertini's theorem,

to the effect that if x, y are two elements of K, algebraically independent over k, and such that $y \notin K^p k$, then for all but a finite number of constants $c \in k$, K is regular over $k(x + cy)$. (Cf. [1], Ch. 6.) If k is finite, one needs a small additional argument. However, in view of Proposition 2, for our purposes, we may assume k infinite.

We have thus reduced the proof of Theorem 1 to the case where k is algebraically closed and K is of transcendence degree 1 over k, i. e. is the function field of a curve.

3. The weak Mordell-Weil Theorem. By a *global field* we shall mean a field K which is either

a function field over an algebraically closed constant field k (i. e. a finitely generated regular extension of k)

or *an algebraic number field of finite degree over the rationals.*

We refer to these as the *function field case* and *number field case* respectively. We let $\dim K$ be the transcendence degree of K over k in the function field case. It is the dimension of a model of K over k. Our aim is to prove Theorems 1 and 2. The results of § 2 (and eventually those of § 8) will be used only to deal with the function field case. The case of number fields is thus somewhat simpler to deal with.

Let m be a natural number prime to the characteristic of K. Let A be an abelian variety defined over K, and let A_m denote the group of points of order m on A, i. e. the kernel of $m\delta$. To prove Theorem 1 or 2, we may assume that all point of A_m are rational over K, because we may deal with the finite separable extension $K(A_m)$ of K instead of K itself. This is obvious in the case of number fields, and follows from Proposition 1 in the case of function fields.

In this section, we prove the weak Mordell-Weil theorem, namely: *If A is an abelian variety defined over a global field K, such that $A_m \subset A_K$, and such that* $\dim K = 1$ *in the function field case, then the factor group A_K/mA_K is finite.* We reproduce essentially word for word the proof given in [3]. Although we could have limited ourselves to making a reference to that paper, we prefer to make our treatment here as self-contained as possible.

We shall use only two elementary and well-known properties of K. To state them, we denote by $\{\mathfrak{p}\}$ the set of all finite primes of K (i. e. in the case of function fields with $\dim K = 1$, the set of points of a model of K/k,

rational over k, and in the case of number fields, the set of all finite primes). Our properties may then be stated as follows.

a.) Let S be a finite set of primes. Then there are only a finite number of abelian extensions of K of exponent m (i.e. such that $\sigma^m = 1$ for all automorphisms of the extension over K) which are unramified outside S.

b.) There exists a finite set S of primes such that for $\mathfrak{p} \notin S$, the abelian variety $A_{\mathfrak{p}}'$ obtained by reducing A modulo $\mathfrak{p}$ is non-degenerate.

For the reduction of varieties, we refer the reader to Shimura [7]. We recall that a reduction (or specialization) of an abelian variety is said to be non-degenerate if the specialized cycle has one component, with multiplicity 1, which is an abelian variety whose law of composition is obtained by specializing that of A. Property b.) may of course also be expressed by saying that in an algebraic system of varieties whose generic member is an abelian variety, almost all members of the system are also abelian varieties whose laws of composition are obtained by specializing that of the generic member. Here, the parameter variety is a model of K/k in the case of function fields, and is a so-called absolute curve in the case of number fields.

The following two lemmas achieve what we want. The first one ties up the factor group A_K/mA_K with the extension $K(1/m \cdot A_K)$ of K obtained by adjoining to K all points $y \in A$ such that $my \in A_K$. It is an abelian extension, whose automorphisms are induced by translations of A_m, in view of the fact that we assumed $A_m \subset A_K$. Indeed, if σ is in the Galois group G and $my = x \in A_K$ then $\sigma(my) = m\sigma y = x$, so that $\sigma y - y \in A_m$.

LEMMA 1. *The factor group A_K/mA_K is finite if and only if $K(1/m \cdot A_K)$ is a finite extension of K.*

Proof. One implication is obvious. Conversely, assume that $K(1/m \cdot A_K)$ is finite over K. We have a bilinear map

$$(A_K, G) \to A_m$$

obtained as follows. Let $x \in A_K$. Let y be such that $my = x$. We put $(x, \sigma) = \sigma y - y$. This element of A_m obviously does not depend on the y selected such that $my = x$. It is clear that (x, σ) is bilinear in x and σ. It is trivially verified that the right-hand kernel of our pairing is 1, while the left-hand kernel is precisely mA_K. The groups A_K/mA_K and G are therefore paired exactly into A_m. Since G and A_m are finite, it follows that A_K/mA_K is also finite.

To conclude the proof of the weak Mordell-Weil Theorem, there remains but to show that $K(1/m\cdot A_K)$ is finite over K. This follows from a.) and b.) and our second lemma.

LEMMA 2. *If $\mathfrak{p}$ is a prime such that $A_{\mathfrak{p}}'$ is non-degenerate, and $\mathfrak{p}\nmid m$, then $K(1/m\cdot A_K)$ is unramified over K at $\mathfrak{p}$.*

Proof. This being a local statement, we may go over to the completion $K_{\mathfrak{p}}$ of K with respect to $\mathfrak{p}$. In the function field case, $K_{\mathfrak{p}}$ is the power series field over k. Let $x\in A_k$. It suffices to show that $K_{\mathfrak{p}}(1/m\cdot x)$ is unramified over $K_{\mathfrak{p}}$. We use a prime to denote the reduction of objects modulo $\mathfrak{p}$. Let $\mathfrak{a}=(m\delta)^{-1}(x)$. Then $\mathfrak{a}'=(m\delta')^{-1}(x')$, since $A_{\mathfrak{p}}'$ is non-degenerate. All points in $(m\delta')^{-1}(x')$ occur with multiplicity 1, since $\mathfrak{p}\nmid m$, and are rational over a finite separable extension L' of $K_{\mathfrak{p}}'$. (Here, $K_{\mathfrak{p}}'$ is the residue class field, and in the function field case, we have $L'=K_{\mathfrak{p}}'=k$.) By a suitable form of Hensel's lemma [3], it follows that all points of $\mathfrak{a}$ are rational over the unramified extension of $K_{\mathfrak{p}}$ obtained by lifting L'. In the case of function fields, this unramified extension is of course $K_{\mathfrak{p}}$ itself.

The needed form of Hensel's lemma, as stated in [3], is the following. *Let L be complete under a discrete valuation, with residue class field L'. Let $\mathfrak{a}$ be a positive 0-cycle, in a projective space, say, rational over L, and let P' be a point of $\mathfrak{a}'$ which is rational over L', and of multiplicity 1 in $\mathfrak{a}'$. Then there is a unique point P in $\mathfrak{a}$ which specializes to P', and P is rational over L.* The proof is immediate, taking into account the uniqueness of the extension of the valuation to algebraic extensions of L.

4. Heights in function fields. In order to carry out the infinite descent in § 7, we need to be able to measure the size of a point rational over K, or as is customary to say, its height. It is convenient to give the definition of the height separately for the function field and number field case. We do this in this section and the next. The fundamental properties of heights are given in § 6, and there the statements and proofs can again be formulated uniformly for the two cases.

In this section, we assume that k is algebraically closed. Let W be a complete non-singular curve defined over k, and let $K=k(w)$ be a function field for W over k, (w) being a generic point of W over k. We follow [6] essentially without change.

We shall define the height of a point P in projective space $\boldsymbol{P}^n$, rational over K. Let $(y_0,\cdots,y_n)$ be a set of homogeneous coordinates for P, rational

over K. Then the y_i may be viewed as functions on W, defined over k. We define the *height of* P to be

$$h(P) = h(y) = -\deg \inf_i (y_i), \tag{1}$$

where (y_i) denotes as usual the divisor of y_i. Since the degree of the divisor of a function is equal to 0, we see on the one hand that $h(P)$ does not depend on the set of homogeneous coordinates representing P, and on the other hand that $h(P) \geqq 0$ (because we could take say $y_0 = 1$). Furthermore, $h(P) = 0$ if and only if P is rational over k, i. e. is constant.

One can give an alternate geometric definition of the height. Let $T(P)$, which we also write $T(y)$, be the locus of P over k. It is a curve, which may of course have singularities. We have a rational map $f\colon W \to \boldsymbol{P}^n$ such that that $f(w) = (y)$. It is induced by a surjective rational map of W onto $T(y)$. We contend that

$$h(y) = (\deg f) \deg T(y), \tag{2}$$

where $\deg f$ is the degree of the rational map of W onto $T(y)$ (non-zero if f is not constant), and $\deg T(y)$ is the projective degree of the variety $T(y)$. To prove this, we may assume without loss of generality that none of the y_i is 0, and that (y) is not constant. Let (Y) be the variables of $\boldsymbol{P}^n$, and let $H = H(Y)$ be a hyperplane defined over k, such that $T(y) \cdot H$ is defined. Put $z = H(y)$. Then for each i, y_i/z is an element of K, and thus a function on W. Furthermore, if we denote by H_i the divisor of the hyperplane $Y_i = 0$, then $H_i - H$ is the divisor of a function on $\boldsymbol{P}^n$, which can also be viewed as a function on $W \times \boldsymbol{P}^n$. Let Γ_f denote the graph of f. It is biholomorphic to W under projection on the first factor. Under this identification, it is clear that the function on $W \times \boldsymbol{P}^n$ whose divisor is $W \times (H_i - H)$ induces y_i/z on Γ_f. Hence by [10], F-VIII$_3$, Th. 4, Cor. 2, we get

$$(y_i/z) = f^{-1}(H_i) - f^{-1}(H).$$

By definition, we have

$$h(y) = -\deg \inf_i (y_i/z) = -\deg \inf_i [f^{-1}(H_i) - f^{-1}(H)].$$

Since the hyperplanes H_i have no point in common, neither do the divisors $f^{-1}(H_i)$ on W, and consequently, $h(y) = -\deg(-f^{-1}(H)) = \deg(f^{-1}(H))$. We have

$$h(y) = \deg[\Gamma_f \cdot (W \times H)]$$

since the degree of a 0-cycle does not change under projection. Projecting on the right, we have

$$h(y) = (\deg f)(\deg T(y)\cdot H)$$

which proves our contention.

From definition (2) of he height, we see that for $h(y)$ bounded, the degree of $T(y)$ is also bounded. Hence by the theory of Chow coordinates, we get our first property of heights.

PROPERTY 1F. *In the function field case, with* $\dim K = 1$, *let* (y) *be a point of* $\mathbf{P}^n$ *rational over* K. *Let* $T(y)$ *be the locus of* (y) *over* k. *If* $h(y)$ *is bounded, then* $\deg T(y)$ *is bounded, and* $T(y)$ *can belong only to a finite number of algebraic families.*

The definition of heights can also be given when $\dim W > 1$ [6]. In fact, let W be a projective normal model of K over k. If $P = (y)$ is again a point in $\mathbf{P}^n$, rational over K, we can define $h(P)$ as in (1), with the understanding that deg now denotes the projective degree of the divisor $\inf_i(y_i)$, in the given projective embedding of W. Thus if $\dim W > 1$, the height of P depends on the choice of model for K over k, while it does not if $\dim W = 1$. As in the case where $\dim W = 1$, we have

PROPOSITION 4. *Let* W *be a projective normal model of* K *over the algebraically closed field* k. *Let* $(y_0,\cdots,y_n)$ *be projective coordinates of a point* P *in* $\mathbf{P}^n$, *with* $y_i \in k(W) = K$. *Let* $T(P)$ *be the locus of* P *over* k, *and* $f\colon W \to \mathbf{P}^n$ *the corresponding rational map of* W *into* $\mathbf{P}^n$. *Then for a generic hyperplane* H *of* $\mathbf{P}^n$, *we have*

$$h(P) = -\deg \inf_i(y_i) = \deg f^{-1}(H).$$

Proof. The arguments follow exactly those given above for curves. The degree is the projective degree in the given projective embedding of W. One sees immediately that the projection on W of any subvariety of codimension 1 of Γ_f must be of codimension 1 on W, and hence simple on Γ_f since f is defined at such a subvariety. Thus W and Γ_f are biholomorphic at such a subvariety, and the arguments given above hold (especially [10], F-VIII$_3$, Th. 4, Cor. 2).

The analogue of (2) if $\dim W > 1$ could also be given, but we omit it. We do not need it for the proof of Theorem 1. We note that the properties of heights proved in § 6 depend only on definition (1).

When dealing simultaneously with the function field and number field

case, it is often convenient to replace the height as we have defined it in (1) by an exponential of it. Indeed, in number fields, the valuations are usually written multiplicatively because of the archimedean ones. To make arguments run completely parallel and to avoid a repetition of formulas in additive and multplicative notation, we therefore define the multiplicative height in function fields as follows. Let c be a number, $0 < c < 1$. We put

$$h^*(P) = c^{-h(P)} \tag{3}$$

and call this the *multiplicative height,* or simply height if it is used constantly throughout a section. Let $\dim W = 1$. For each point $\mathfrak{p}$ of W, rational over k, and a function $y \in K$, we define the absolute value $v_{\mathfrak{p}}$ as usual by

$$v_{\mathfrak{p}}(y) = c^{\operatorname{ord}_{\mathfrak{p}} y}, \tag{4}$$

where $\operatorname{ord}_{\mathfrak{p}} y$ is the order of the zero of y at $\mathfrak{p}$ (negative if y has a pole). Thus a function has a zero of high order at $\mathfrak{p}$ if $v_{\mathfrak{p}}(y)$ is close to 0, and a pole of high order at $\mathfrak{p}$ if $v_{\mathfrak{p}}(y)$ is large, close to $+\infty$.

In view of the — sign in (1) and the fact that $0 < c < 1$, we get by a trivial computation

$$h^*(P) = \prod_{\mathfrak{p}} \sup_i v_{\mathfrak{p}}(y_i). \tag{5}$$

It is this expression which is used to define the height in number fields, as we shall see in the next section. We note that $h^*(P) \geqq 1$.

If $\dim W > 1$, we shall say that $\mathfrak{p}$ is a *prime divisor of* K if it is a prime divisor of W, i.e. a subvariety of W of codimension 1, defined over k. Then $\deg \mathfrak{p}$ is the projective degree of $\mathfrak{p}$ in the given projective embedding of W, and $v_{\mathfrak{p}}(y)$ is defined by

$$v_{\mathfrak{p}}(y) = c^{(\deg \mathfrak{p}) \operatorname{ord}_{\mathfrak{p}} y}.$$

Formula (5) is then clearly applicable without change.

5. Heights in number fields. Let K be a number field of finite degree over the rationals $\boldsymbol{Q}$. Let $\mathfrak{p}$ be a finite prime of K, corresponding to a prime ideal in its ring of integers. Let $N\mathfrak{p}$ be as usual the number of elements in the residue class field. For $y \in K$, we let

$$v_{\mathfrak{p}}(y) = (1/N\mathfrak{p})^{\operatorname{ord}_{\mathfrak{p}} y},$$

where $\operatorname{ord}_{\mathfrak{p}} y$ is the order of y at the discrete valuation of K determined by $\mathfrak{p}$.

If $\mathfrak{p}$ is an infinite prime, i.e. is a real or a pair of complex conjugate

embeddings of K into the complex numbers, then we let $v_{\mathfrak{p}}(y)$ be the ordinary absolute value if the embedding is into the reals, and the square of the ordinary absolute value if the embedding is not real. The product formula states that for $y \in K$, $y \neq 0$, we have

$$\prod_{\mathfrak{p}} v_{\mathfrak{p}}(y) = 1.$$

Let $(y_0, \cdots, y_m)$ be a set of homogeneous coordinates for a point P in $\boldsymbol{P}^m$, with $y_i \in K$. We define the *height* $h(P)$ of P by the relation

$$h(P)^{[K:\boldsymbol{Q}]} = \prod_{\mathfrak{p}} \sup_i v_{\mathfrak{p}}(y_i). \tag{6}$$

The product formula guarantees that this depends only on the point in projective space and not on the coordinates chosen. Since one of the y_i could have been chosen equal to 1, we see that $h(P) \geqq 1$. The extra term $[K:\boldsymbol{Q}]$ has been added to insure that $h(P)$ does not depend on the field K over which P is rational.

If σ is an automorphism of K over $\boldsymbol{Q}$, then by transport of structure, we have $h(P^\sigma) = h(P)$.

The (absolute) degree $d(P)$ of P is defined to be the degree $[K:\boldsymbol{Q}]$, where $K = \boldsymbol{Q}(P)$ is the field obtained by adjoining to $\boldsymbol{Q}$ a set of affine coordinates for P, say $y_1/y_0, \cdots, y_m/y_0$. We have the following strong version of Property 1F.

PROPERTY 1N. (Northcott) *Let h_0, d_0 be two fixed numbers. Then there is only a finite number of points P in $\boldsymbol{P}^m$ such that $d(P) \leqq d_0$ and $h(P) \leqq h_0$.*

Proof. Let us first consider the case where $d(P) = 1$, i.e. P is rational over $\boldsymbol{Q}$. Multiplying the y_i by a suitable integer, we may assume that all y_i are integers, and that their g. c. d. is equal to 1. For all finite primes p of $\boldsymbol{Q}$, we then get $\sup_i v_p(y_i) = 1$. The height of P is then determined by the ordinary absolute value, and the result is obvious.

We prove the property in general by reducing it to the preceding case.

Let $(T_0, \cdots, T_m)$ be variables. Let

$$F(T) = \sum x_\alpha M_\alpha(T)$$

be a form in the T's, where the M_α are monomials, whose coefficients x_α lie in a number field. Then (x) may be viewed as a point in some projective space $\boldsymbol{P}^N$. We define the *height of F* to be the height of (x), and write it $h(F)$.

LEMMA. *Let d_1, d_2 be two natural numbers. Then there exists a number s depending only on d_1, d_2 such that if F_1, F_2 are two forms of degrees d_1, d_2 respectively with coefficients in a number field K, then*

$$h(F_1F_2) \leqq sh(F_1)h(F_2).$$

Proof. Let $F_1 = \sum x_\alpha M_\alpha(T)$ and $F_2 = \sum y_\beta M_\beta(T)$. Put $F_1F_2 = \sum z_\lambda M_\lambda(T)$. Then

$$z_\lambda = \sum x_{\alpha(\lambda)} y_{\beta(\lambda)},$$

where $\alpha(\lambda)$, $\beta(\lambda)$ range over those indices α, β such that $M_{\alpha(\lambda)}M_{\beta(\lambda)} = M_\lambda$. If $\mathfrak{p}$ is a finite prime of K, then obviously

$$v_\mathfrak{p}(z_\lambda) \leqq \sup_\alpha v_\mathfrak{p}(x_\alpha) \sup_\beta v_\mathfrak{p}(y_\beta),$$

and hence we can add a $\sup_\lambda$ to the left hand side of this inequality.

If $\mathfrak{p}$ is archimedean, then there obviously exists an integer $s_\lambda(d_1, d_2)$ such that

$$v_\mathfrak{p}(z_\lambda) \leqq s_\lambda(d_1, d_2) \sup_\alpha v_\mathfrak{p}(x_\alpha) \sup_\beta v_\mathfrak{p}(y_\beta).$$

This $s_\lambda(d_1, d_2)$ is the number of terms in the sums expressing z_λ in terms of $x_{\alpha(\lambda)} y_{\beta(\lambda)}$, or its square if $\mathfrak{p}$ is complex. Hence there exists an integer $s(d_1, d_2)$ such that

$$\sup_\lambda v_\mathfrak{p}(z_\lambda) \leqq s(d_1, d_2) \sup_\alpha v_\mathfrak{p}(x_\alpha) \sup_\beta v_\mathfrak{p}(y_\beta).$$

Taking the product over all $\mathfrak{p}$, we get

$$h(z)^{[K:Q]} \leqq s(d_1, d_2)^r h(x)^{[K:Q]} h(y)^{[K:Q]},$$

with an integer $r \leqq [K:\boldsymbol{Q}]$. This proves our lemma.

To conclude the proof of Property 1N, let $(y_0, \cdots, y_m)$ be a point of degree d_0, with say $y_0 = 1$. Let

$$F(T) = \prod_\sigma (y_0{}^\sigma T_0 + \cdots + y_m{}^\sigma T_m)$$

the product being taken over all conjugates of the field $\boldsymbol{Q}(y)$, so that $F(T)$ is of degree d_0. We call $F(T)$ the *Chow form* of $P = (y)$. It is clear that two points have the same Chow form if and only if they are conjugate over $\boldsymbol{Q}$, because $F(T)$ factorizes essentially uniquely. Hence by the lemma, we get

$$h(F) \leqq s(d_0) h(P)^{d_0}, \tag{7}$$

where $s(d_0)$ is an integer depending only on d_0, and not on P. Our property is obvious from this relation, the first paragraph of the proof, and the fact that $h(P) = h(P^\sigma)$.

6. Properties of heights. Let K be a global field, and let V be an abstract variety defined over K. Let λ, λ' be two real-valued functions on V_K (the rational points of V in K). We shall say that they are *equivalent*, and write $\lambda \sim \lambda'$, if there exist two real numbers $c_1, c_2 > 0$ such that for all $P \in V_K$, we have

$$c_1\lambda(P) \leqq \lambda'(P) \leqq c_2\lambda(P).$$

This equivalence relation will be applied to heights. In this section and the next, the height is taken to be multiplicative in the function field case, and we write h instead of h^*. (If it were taken additive, as in (1), then we would make the above relation read

$$c_1 + \lambda(P) \leqq \lambda'(P) \leqq c_2 + \lambda(P)$$

and we would omit the condition $c_1, c_2 > 0$.)

Actually, we remark (as in [8]) that in number fields, we can consider the stronger equivalence relation where λ, λ' are functions on all absolutely algebraic points of V, the constants working uniformly for all such points. All the statements which we shall make concerning the equivalence of heights in the sequel are valid under this stronger interpretation. If we adopted a device in function fields analogous to the one in number fields, i.e. once K is fixed, define the height for points rational over the algebrais closure $\bar{K}$ of K by taking suitable roots, then a similar remark would apply in function fields.

Let now V be a complete, abstract, normal variety defined over K. Let ϕ: $V \to \boldsymbol{P}^m$ be an everywhere defined rational map of V into projective space, defined over K. Then for each point P of V_K, $\phi(P)$ is a point of $\boldsymbol{P}^m$, rational over K, and we can thus define its height, which we shall denote by $h_\phi(P)$. We shall give below conditions under which the real-valued functions h_ϕ are equivalent. As mentioned before, the properties of heights in the case of function fields depend only on definition (1) or (3), and for the proof of Theorem 1, are needed only in the case $\dim K = 1$.

It follows immediately from the definition of the height that if ϕ' is another rational map of V into $\boldsymbol{P}^m$ which differs from ϕ by a projective transformation defined over K, then $h_\phi \sim h_{\phi'}$. Indeed, the elements of K which enter into such a transformation can introduce only a uniformly bounded change in the height of a point because they are fixed once for all.

We shall obtain mappings ϕ: $V \to \boldsymbol{P}^m$ by means of linear systems. Let $\mathfrak{L}$ be a linear system on V, defined over K. This means that we can find a divisor $X_0 \in \mathfrak{L}$, rational over K, such that the space of functions L_0 con-

sisting of all functions on V whose divisors are of type $X - X_0$ with $X \in \mathfrak{L}$ has a basis defined over K. If $(f_0 = 1, \cdots, f_m)$ is such a basis, then it defines a rational map $\phi\colon V \to \boldsymbol{P}^m$. If $\mathfrak{L}$ is without fixed point, then ϕ is everywhere defined ([1], Ch. 6). In view of the remark in the preceding paragraph, we see that the equivalence class of h_ϕ actually depends only on the linear system.

Let $\mathfrak{M}$ be another linear system on V, also defined over K. Then we can define the sum $\mathfrak{L} + \mathfrak{M}$ as a linear system in the usuall manner. If $(f_0 = 1, \cdots, f_m)$ is a basis for L_0 over K, and $(g_0 = 1, \cdots, g_n)$ is a basis for M_0 over K obtained in a similar manner, then we get a vector space of functions N_0 generated by the products $f_i g_j$. We have $f_0 g_0 = 1$. The divisor of $f_i g_j$ is $(f_i g_j) = (f_i) + (g_j)$. If we write $(f_i) = X_i - X_0$ and $(g_j) = Y_j - Y_0$ then $(f_i g_j) = X_i + Y_j - (X_0 + Y_0)$. The space N_0 gives rise to a linear system $\mathfrak{N}$ called the sum of $\mathfrak{L}$ and $\mathfrak{M}$. If $\mathfrak{L}$ and $\mathfrak{M}$ are both without fixed points, so is $\mathfrak{L} + \mathfrak{M}$.

Let $\phi\colon V \to \boldsymbol{P}^m$ and $\psi\colon V \to \boldsymbol{P}^n$ be two rational maps derived from $\mathfrak{L}$ and $\mathfrak{M}$. The functions $\{f_i g_j\}$ give rise to a rational map η of V into $\boldsymbol{P}^{(m+1)(n+1)-1}$. If we denote by $\phi + \psi$ any one of the rational maps (defined over K) derived from $\mathfrak{L} + \mathfrak{M}$, and determined only up to a projective transformation, then obviously, $h_\eta = h_\phi h_\psi$ and $h_\eta \sim h_{\phi+\psi}$. Summarizing, we get

PROPERTY 2. *Let V be a complete, abstract, normal variety defined over K. Let $\mathfrak{L}$, $\mathfrak{M}$ be two linear systems on V, also defined over K, and without fixed points. Let ϕ, ψ be two rational maps of V into $\boldsymbol{P}^m$, $\boldsymbol{P}^n$ respectively, defined by these systems over K, and let $\phi + \psi$ be a rational map defined over K by $\mathfrak{L} + \mathfrak{M}$. Then $h_{\phi+\psi} \sim h_\phi h_\psi$.*

The next property also follows immediately from the definitions.

PROPERTY 3. *Let U, V be two complete, abstract varieties defined over K. Assume U normal and V non-singular. Let $\omega\colon U \to V$ be an everywhere defined, surjective rational map, defined over K. Let $\mathfrak{L}$ be a linear system on V, defined over K, and let $\omega^{-1}(\mathfrak{L})$ be the linear system on U consisting of all divisors $\omega^{-1}(X)$ as X ranges over $\mathfrak{L}$. Let ϕ be a rational map of V into $\boldsymbol{P}^m$ defined by $\mathfrak{L}$ over K. Then $\phi \circ \omega$ is a rational map associated with the linear system $\omega^{-1}(\mathfrak{L})$. Assume in addition that $\mathfrak{L}$ is without fixed points. Then so is $\omega^{-1}(\mathfrak{L})$, and we have*

$$h_{\phi \circ \omega} = h_\phi \circ \omega.$$

Proof. We have assumed V non-singular in order to insure that the

inverse image of a divisor linearly equivalent to 0 is also linearly equivalent to 0 (cf. [2], App. 1). That $\omega^{-1}(\mathfrak{L})$ is then a linear system is obvious, and so is the rest of our assertions.

The preceding three properties of heights have been trivial consequences of the definitions. Our fourth and last property, although not difficult to prove, will require a slightly more elaborate argument. We have taken it and its proof from Weil's paper [8].

PROPERTY 4. *Let V, $\mathfrak{L}$, $\mathfrak{M}$, ϕ, ψ be as in Property 2. If the divisors of $\mathfrak{L}$ and $\mathfrak{M}$ are linearly equivalent to each other, then $h_\phi \sim h_\psi$.*

Proof. In the course of the proof we shall use frequently the fact that if a function $f \in K(V)$ is not defined at a point Q of V, then it has a pole passing through Q ([1], Ch. 6) and hence if there exists a place of $K(V)$ over K which maps f on ∞ (resp. on 0), then f has a pole (resp. a zero) passing through Q.

By transitivity, we may clearly assume that $\mathfrak{M}$ is the complete linear system containing $\mathfrak{L}$.

Let X_0 be as before a divisor of $\mathfrak{L}$, rational over K, such that the derived space of functions L_0 has a basis $(f_0, \cdots, f_m)$ defined over K. For each f_i we can write

$$(f_i) = X_i - X_0.$$

We denote by ϕ_i the rational map of V into the affine K-open subset of $\boldsymbol{P}^m$ determined by the functions $(f_0/f_i, \cdots, f_m/f_i)$. We let V_i be the K-open subset $V - \operatorname{supp}(X_i)$ of V. Then the V_i cover V, and ϕ_i is defined at every point of V_i.

In view of our assumption on $\mathfrak{M}$, we may take $Y_0 = X_0$ and $M_0 = L(X_0)$. Let $(g_0, \cdots, g_n)$ be a basis of $L(X_0)$ defined over K, and put

$$(g_j) = Y_j - X_0.$$

We may assume that $g_i = f_i$ $(i = 0, \cdots, m)$. We have a rational map ψ_j of V into the affine K-open subset of $\boldsymbol{P}^n$ determined by $(g_0/g_j, \cdots, g_n/g_j)$, and ψ_i is defined at every point of V_i for $i = 0, \cdots, m$. Thus to compute $h_\psi(P)$ for $P \in V_K$, we may restrict ourselves to $h_{\psi_i}(P)$ for $P \in V_i \cap V_K$.

Consider first the points of V_K which are in V_0. Let g be any one of the g_j $(j = 0, \cdots, n)$. We contend that the ideal generated by $(f_0/g, \cdots, f_m/g)$ in the ring $K[f_0/g, \cdots, f_m/g]$ is the unit ideal. Otherwise, it is contained in a maximal ideal, and there is a place of the function field $K(V)$ over K which maps all f_i/g on 0. This place induces a point Q on V, and hence

the functions f_i/g must all have a zero passing through Q. Since $(f_i/g) = X_i - Y$ with $Y > 0$, this implies that $Q \in \operatorname{supp}(X_i)$ for all i, and contradicts our assumption that they have no point in common.

From the above, we can write

$$1 = \sum z_\nu \boldsymbol{M}_\nu(f_i/g)$$

with $z_\nu \in K$. Here, $\boldsymbol{M}_\nu$ stands for a monomial $(f_0/g)^{\nu_0} \cdots (f_m/g)^{\nu_m}$. Note that the z_ν do not depend on $P \in V$, and that $\deg \boldsymbol{M}_\nu \geqq 1$.

Put $x_i = f_i(P)$, where P is any point in V_0, and $y = g(P)$. Suppose first that $y \neq 0$. Then

$$1 = \sum z_\nu \boldsymbol{M}_\nu(x_i/y).$$

For every prime $\mathfrak{p}$ of K, it follows that there exists a number $c_\mathfrak{p} > 0$ which is equal to 1 for all but a finite number of $\mathfrak{p}$, such that

$$\sup_i v_\mathfrak{p}(x_i/y) \geqq c_\mathfrak{p}.$$

(In function fields, this means that there exists a finite set S of prime divisors of W such that the x_i/y cannot have a common zero for $\mathfrak{p} \notin S$, and that for $\mathfrak{p} \in S$, the order of such a common zero cannot be arbitrarily high. The set S can be taken to be the set where some z_ν has a pole.)

Since we selected g arbitrarily among the g_j, we get

$$\sup_i v_\mathfrak{p}(x_i) \geqq c_\mathfrak{p} \sup_j v_\mathfrak{p}(y_j),$$

where $y_j = g_j(P)$. We emphasize that $c_\mathfrak{p}$ depends only on the z_ν, and not on $P \in V_0 \cap V_K$. Furthermore, this formula clearly holds whether $y_j = 0$ or not, i.e. for all P in $V_0 \cap V_K$. Taking the product, we see that there exists a number $c_1 = c_1(V_0) > 0$ such that for all $P \in V_0 \cap V_K$ we have

$$c_1 h_\psi(P) \leqq h_\phi(P).$$

By symmetry, using the same arguments as above, there exists a constant $c_2 > 0$ such that for all $P \in V_0 \cap V_K$, we have

$$c_1 h_\psi(P) \leqq h_\phi(P) \leqq c_2 h_\psi(P).$$

Since V is covered by the finite number of K-open sets $V_0, \cdots, V_m$, we can repeat the above procedure for each one of them. We thus obtain numbers $c_1(V_i)$ and $c_2(V_i)$. In the statement of our property, we simply take the smallest of the former and the largest of the latter to conclude the proof.

7. The infinite descent. Let K be a global field, and let A be an abelian variety embedded in a projective space $\boldsymbol{P}^n$ over K. The height of a point $P \in A_K$ determined by the identity mapping of A into $\boldsymbol{P}^n$ is denoted by $h(P)$. In the function field case, we deal with the multiplicative height.

Let $m > 1$ be a natural number such that A_K/mA_K is finite, and let $a_1, \cdots, a_s$ be representatives of A_K/mA_K. Then any point P in A_K is congruent to some a_i mod mA_K. Denote by S a sequence $\{P_0, P_1, \cdots\}$ of points of A_K, constructed by starting with an arbitrary point P_0, and such that

$$mP_{\nu+1} = P_\nu - a_{i_\nu}.$$

We contend that there is a number $c_1 > 0$ independent of S, and an integer $\nu(S)$ depending on S such that for any sequence S, we have $h(P_\nu) \leqq c_1$ for $\nu > \nu(S)$. This contention is an obvious consequence of the fact, to be proved below, that there exists a number $c_2 > 0$ independent of S such that

$$h(P_{\nu+1})^{m^2-1} \leqq c_2 h(P_\nu). \tag{8}$$

Actually, we shall prove

PROPOSITION 5. *Let K be a global field. Let A be an abelian variety embedded in projective space $\boldsymbol{P}^n$ over K. Let $m > 1$ be a natural number, and let $a \in A_K$. Then there exists a number $c_2(a, m)$ depending on a and on m, such that for any $P \in A_K$, we have*

$$h(P)^{m^2-1} \leqq c_2(a, m) h(mP + a).$$

To deduce (8) from Proposition 5, we let a range over the a_i, and let $c_2 = \sup_i c_2(a_i)$.

From the manner in which the sequences are constructed, we see that for each ν, there exist integers $n_1, \cdots, n_s$ such that

$$P_0 = m^\nu P_\nu + n_1 a_1 + \cdots + n_s a_s$$

and we obtain the pay-off of our infinite descent.

COROLLARY. *Let m be such that A_K/mA_K is finite, and let $a_1, \cdots, a_s$ be representatives in A_K of A_K/mA_K. There exist a number c_1 and a subset $\mathfrak{S}$ of A_K such that:*

(i) *$h(P) \leqq c_1$ for all $P \in \mathfrak{S}$, and*

(ii) *for any $P_0 \in A_K$, there exist integers $n_0, n_1, \cdots, n_s$ and a point $P \in \mathfrak{S}$ such that*

$$P_0 = n_0 P + n_1 a_1 + \cdots + n_s a_s.$$

In the case of number fields, we apply Property 1N to conclude the proof of Theorem 2. The case of function fields will require an additional argument, which will be supplied in the next and final section.

We observe that A_K/mA_K was proved finite in § 3 only for a special case of K. A posteriori, once Theorems 1 and 2 are proved, one sees of course that A_K/mA_K is finite for all global fields.

Let us now prove Proposition 5. Let a be a point of A_K. Let $\omega\colon A \to A$ be the rational map $\omega u = mu + a$. Let X be a hyperplane section of A in its given projective embedding, rational over K. From now on we use constantly the results and notations of [2], Ch. 5 concerning divisors on abelian varieties. Aside from those, we use only Properties 2, 3, 4 of heights. To begin with, we have immediately from the definitions

$$\omega^{-1}(X) = (m\delta)^{-1}(X_{-a}) = (m\delta)^{-1}(X_{-a} - X) + (m\delta)^{-1}(X).$$

We also have

$$(m\delta)^{-1}(X) \equiv m^2 X,$$

this being deeper. One knows that the equivalence $\equiv$ is the same as algebraic equivalence. (This uses the fact that there is no torsion, not proved in [2]. Using only what is proved there, namely that it is the same as the torsion equivalence, we could dispense with this, at the cost of introducing in an obvious manner a multiple of the equivalences below.) Since X is a hyperplane section, the homomorphism $u \to \mathrm{Cl}(X_u - X)$ of A into $\hat{A}$ is surjective. Hence there is a point b of A such that

$$(m\delta)^{-1}(X) - m^2 X \sim X_b - X.$$

Furthermore, for the same reason, there exists a point $c \in A$ such that $(m\delta)^{-1}(X_{-a} - X) \sim X_c - X$. Thus we get finally

$$\begin{aligned} \omega^{-1}(X) &\sim X_c - X + m^2 X + X_b - X \\ &\sim m^2 X + X_d - X \\ &\sim (m^2 - 1)X + X_d \end{aligned}$$

with $d = b + c$. Now d is not necessarily rational over K, but since X is ample, so is X_d. Since $\omega^{-1}(X) - (m^2 - 1)X$ is linearly equivalent to an ample divisor over some field, and is itself rational over K, we conclude that there exists an ample positive divisor X' rational over K such that

$$\omega^{-1}(X) \sim (m^2 - 1)X + X'. \tag{9}$$

The divisors in the complete linear system containing $\omega^{-1}(X)$ are linearly

equivalent to the divisors of the sum of the complete linear systems containing $(m^2-1)X$ and X'. If $Y>0$ is a divisor on A rational over K, whose complete linear system $\mathfrak{L}(Y)$ is without fixed point, we select a definite rational map ϕ_Y defined over K, associated with $\mathfrak{L}(Y)$, in the class of such mappings determined only up to a projective transformation. We shall write h_Y instead of h_{ϕ_Y}. From (9) and Properties 2 and 4 we get

$$h_{\omega^{-1}(X)} \sim h_X{}^{(m^2-1)} h_{X'}.$$

Also by Property 4, we have $h \sim h_X$. (The linear system of hyperplane sections in the given embedding of A in $\boldsymbol{P}^n$ might not be complete, so we cannot write an equality.) Hence by Property 3 and 4 we see that there exists a number $c_2 = c_2(a, m)$ such that

$$h(P)^{m^2-1} h_{X'}(P) \leqq c_2 h(\omega(P)).$$

From the definition of the height (multiplicative in function fields), we know that $h_{X'}(P) \geqq 1$ for all $P \in A_K$. Hence

$$h(P)^{m^2-1} \leqq c_2 h(mP+a).$$

This concludes the proof of Proposition 5, and of Theorem 2.

8. End of the proof of Theorem 1. We consider the function field case. We let $W \subset \boldsymbol{P}^r$ be a projective, normal variety defined over the algebraically closed field k. Let w be a generic point of W over k, and $K = k(w)$ a function field for W over k. Let A be an abelian variety defined over K, and embedded in projective space $\boldsymbol{P}^n$ over K. Let u be a generic point of A over K. We shall denote by $W \circ A$ the locus of (w, u) over k. Then $W \circ A$ is the graph of the algebraic family parametrized by W, of which A is a generic member. (If we put $Z = W \circ A$, then $A = Z(w) = \mathrm{pr}_2[Z \cdot (w \times \boldsymbol{P}^n)]$.)

Given a rational point P of A over K, we shall denote by W_P the locus of (w, P) over k. Then

$$W_P \subset W \circ A \subset W \times \boldsymbol{P}^n \subset \boldsymbol{P}^r \times \boldsymbol{P}^n.$$

Note that W_P and W are birationally equivalent under the projection pr_1 (biholomorphic if W is a curve).

There is a projective embedding $\phi\colon \boldsymbol{P}^r \times \boldsymbol{P}^n \to \boldsymbol{P}^{(r+1)(n+1)-1}$ which to each products of points with homogeneous coordinates $(x_0, \cdots, x_r) \times (y_0, \cdots, y_n)$ assigns the point with homogeneous coordinates $(x_i y_j)$. One verifies immediately that for any $P \in A_K$, we have

$$h_\phi(w, P) \leqq h(w) + h(P) = \deg W + h(P).$$

Assume $\dim K = 1$. If $\mathfrak{S}$ is a subset of A_K such that $h(P) \leqq c_3$ (the constant of the corollary to Proposition 5), then by Property 1F, there exists a constant c_4 such that $\deg \phi(W_P) \leqq c_4$, and hence W_P can belong only to a finite number of algebraic families on $W \circ A$ for $P \in \mathfrak{S}$. In view of the corollary to Proposition 5, we shall be through with the proof of Theorem 1 once we have proved

THEOREM 3. *Let W be a projective, normal variety defined over the algebraically closed field k. Let w be a generic point of W over k, and $K = k(w)$ a function field of W over k. Let A be an abelian variety defined over K, and let (B, τ) be its K/k-trace. For each point $P \in A_K$, let W_P be the locus of (w, P) over k. If P, $P' \in A_K$ are such that W_P and $W_{P'}$ are in the same algebraic family on $W \circ A$, then P and P' are congruent modulo ${}_\tau B_k$* (i.e. $P - P' \in {}_\tau B_k$).

Proof. We shall prove our theorm by going from W_P to $W_{P'}$ by passing through a generic member of the family. Our first step is to show that a generic member is of the same type as W_P.

LEMMA. *Let U be a subvariety of $W \circ A$, defined over an extension k^* of k, independent of K over k, and such that W_P is a specialization of U over k. Put $K^* = Kk^*$. Then there exists a rational point P^* of A in K^* such that $U = W_{P^*}$.*

Proof. We have $\mathrm{pr}_1 W_P = W$. Hence by the compatibility of projections with specializations [4], we must have $\mathrm{pr}_1 U = W$. But $\dim U = \dim W$, and hence a generic point of U over k^* is of type (w, Q) for some $Q \in \boldsymbol{P}^n$, algebraic over $k^*(w)$. We must show that Q is rational over $k^*(w)$, and that Q is in A.

On $W \times \boldsymbol{P}^n$ the intersection $U \cdot (w \times \boldsymbol{P}^n)$ is obviously defined, and in view of the above remarks, we have

$$U \cdot (w \times \boldsymbol{P}^n) = \sum_{i=1}^{d} (w, Q_i)$$

for suitable points Q_i. Taking the projection on W, we see immediately that $d = 1$. Since $U \cdot (w \times \boldsymbol{P}^n)$ is rational over $k^*(w)$, so is $Q_1 = P^*$. Finally, we have $P^* \in A$ because

$$w \times P^* = U \cdot (w \times \boldsymbol{P}^n) \subset (W \circ A) \cdot (w \times \boldsymbol{P}^n) = w \times A.$$

Note that in the lemma, we have made no use of the fact that A is an abelian variety: The same statement holds for any algebraic system of varieties.

It will suffice to prove Theorem 3 for W_P and W_{P^*}. Indeed, given W_P and $W_{P'}$ as in the theorem, there exists a subvariety W_{P^*} of $W \circ A$, defined over some k^*, such that both W_P and $W_{P'}$ are specializations of W_{P^*}. If $P - P^* \in {}_T B_{k^*}$ and $P' - P^* \in {}_T B_{k^*}$, then $P - P' \in {}_T B_{k^*}$, and by Proposition 2, $P - P' \in {}_T B_k$.

The field k^* may also be assumed to be of transcendence degree 1 over k, i.e. the parameter variety in the algebraic system joining W_P to W_{P^*} may be assumed to be a curve, by blowing up the point corresponding to W_P on the given parameter variety joining W_P and W_{P^*}. Algebraically speaking, there exists a field $k_1 \supset k$, a complete non-singular curve T defined over k_1, a generic point t^* of T over k_1, and a point t of T, rational over k_1, such that if we put $k^* = k_1(t^*)$, the variety W_{P^*} is defined over k^*, and W_P is the unique specialization of W_{P^*} over $t^* \to t$ (relative to k_1) [4]. Without loss of generality for the proof of Theorem 3, we may let $k = k_1$ (using once more Proposition 2). We have the following diagram of fields:

$$\begin{tikzcd}
 & K^* = k^*(w) \arrow[dl, dash] \arrow[dr, dash] & \\
K = k(w) \arrow[dr, dash] & & k^* = k(t^*) \arrow[dl, dash] \\
 & k &
\end{tikzcd}$$

Let $\mathrm{Alb}(T)$ be the Albanese variety of T, defined over k, and let $f: T \to \mathrm{Alb}(T)$ be a canonical map, defined over k. Let $g: T \to A$ be the rational map defined over K by the expression $g(t^*) = P^*$. Then there is a homomorphism $\alpha: \mathrm{Alb}(T) \to A$ and a point $a \in A$ such that the following diagram is commutative.

$$\begin{tikzcd}
T \arrow[r, "f"] \arrow[d, "g"'] & \mathrm{Alb}(T) \arrow[dl, "\alpha + a"] \\
A &
\end{tikzcd}$$

By the definition of the K/k-trace, there exists a homomorphism $\beta: \mathrm{Alb}(T) \to B$ defined over k such that the following diagram is commutative.

$$\begin{tikzcd} & \mathrm{Alb}(T) \arrow[dl, "\alpha"'] \arrow[d, "\beta"] \\ A & B \arrow[l, "\tau"] \end{tikzcd}$$

Since t is simple on T, f is defined at t. Furthermore, since W_P is the unique specialization of W_{P^*} over $t^* \to t$ (relative to k or K), it follows that P is the unique specialization of P^* over $t^* \to t$ (relative to K). Hence $g(t) = P$, and

$$\begin{aligned} P^* - P = g(t^*) - g(t) &= \alpha f(t^*) + a - \alpha f(t) - a \\ &= \tau\beta f(t^*) - \tau\beta f(t) \\ &= \tau b \end{aligned}$$

with $b = \beta[f(t^*) - f(t)] \in B_{k^*}$. QED.

PARIS.

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ON GROUPS OF MEASURE PRESERVING TRANSFORMATIONS. I.*

By H. A. Dye.[1]

1. Introduction. This study concerns the classification of automorphism groups of a finite non-atomic measure algebra, the object being the description of equivalence classes of groups under certain rather pervasive notions of equivalence. In essence, these are the following. Given a group G of measure preserving automorphisms α of the finite non-atomic measure algebra (M, λ), one considers those automorphisms α of (M, λ) which depend locally on G, or more precisely, which have a representation

$$\alpha(P) = \sum_n Q_n \alpha_n(P), \tag{1.1}$$

where the α_n are elements of G, the Q_n are mutually disjoint elements of M, and $\sum$ denotes the least upper bound in (M, λ). The collection of all automorphisms of (M, λ) which depend locally on G forms a group $[G]$ containing G, called the full group determined by G. Two groups G_1 and G_2 are called equivalent if they determine the same full group, and weakly equivalent if there exists an isomorphism φ between the algebras on which they act such that G_1 and $\varphi^{-1}G_2\varphi$ are equivalent. The effect of these equivalence concepts is to erase many distinctions of significance in traditional ergodic theory; for example, it develops that any two singly-generated ergodic automorphism groups acting on separable non-atomic measure algebras are weakly equivalent. Attention shifts, rather, to the local properties of automorphism groups: given a group G and an element P of M, we denote by $[G]_P$ the local subgroup of $[G]$ consisting of all automorphisms in $[G]$ which act as the identity off P, and say that a property of G is local if its validity for $[G]_P$ entails its validity for $[G]_{\bar{P}}$, where $\bar{P}$ denotes the smallest G-invariant element of M containing P. Three basic local properties are analyzed in this paper, concepts of type (I and II), and a concept of approximate finiteness.

Any automorphism group G splits as the direct sum of a group of type I and a group of type II. Groups of type I are uniquely decomposable, up to equivalence, into direct sums of freely-acting cyclic groups of finite order. Groups of type II—by definition, those with no type I summands—comprise,

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therefore, the interesting class. These abound in practice, any freely-acting infinite group being of type II, and are presumably highly ramified under equivalence. The heuristic viewpoint is that "type II" embraces a class of local properties, each strong enough to assure weak equivalence of all groups possessing such a property and having isomorphic fixed algebras (this, under appropriate homogeneity assumptions). Unfortunately, approximate finiteness, which we discuss below, is the only such local property known at present. And because implications of this property are not yet fully understood—the question arises, for example, if the approximate finiteness of a group does not depend on its algebraic structure alone, and not on its specific realizations as automorphism groups—further detailed study of approximate finiteness seems desirable for clues on the general type II structure.

By definition, an automorphism group G of (M, λ) is approximately finite if elements in each finite subset of G can be approximated simultaneously by elements in a finite subgroup of $[G]$. More precisely, given an arbitrary finite subset $\alpha_1, \cdots, \alpha_n$ of G and an $\epsilon > 0$, one requires that there exist a finite subgroup K of $[G]$ and elements $\alpha'_1, \cdots, \alpha'_n$ of K such that $\alpha_i(P) = \alpha'_i(P)$, for all P dominated by some E_i in M with $\lambda(I - E_i) < \epsilon$. A basic theorem in the subsequent development asserts that any countably generated approximately finite group is equivalent to a direct product of finite cyclic groups; and an alternative presentation of the theory can be made in which this statement serves as initial definition. Our main results on approximately finite groups are the following: any singly generated automorphism group is approximately finite (Theorem 1); any subgroup of an approximately finite group is approximately finite (Theorem 2); if M is countably generated over the fixed algebra Z of an automorphism group G, then G contains a maximal approximately finite subgroup with fixed algebra Z (Theorem 4); and finally, under certain countability assumptions, two approximately finite groups of type II are weakly equivalent if and only if they have isomorphic fixed algebras (Theorem 5). The existence of non-approximately finite groups is established by example in Section 8.

It will be evident to the specialist in operator theory that these results, after Theorem 2, correspond precisely to main theorems in the Murray-von Neumann-Kaplansky theory of approximately finite W^*-algebras of type II (see [2], [5], [8]). The present theory evolved, in fact, from a study of regular maximal abelian W^*-subalgebras of finite W^*-algebras, in the course of which this measure theoretic prototype of such algebras was detected. More than analogy is involved, since non-commutative integration techniques enable

one to translate the results in this measure theoretic model directly to operator theory, and to subsume thereby important aspects of operator algebra theory, this being the case, for example, with much of the Murray-von Neumann theory of approximately finite factors. Many of the stumbling blocks which beset the specialist in operator algebras, to cite non-approximately finite II_1's, are stumbling blocks in this measure theoretic model; but this model has the virtue of much greater technical simplicity, and seems the better place to confront certain unsolved problems. For the sake of unity, explicit discussion of these connections with operator theory is not undertaken in this paper. Such discussion, together with further development of the present theory itself, including non-finite measure algebras, will be given elsewhere.

2. Technical preliminaries. Any finite measure algebra is the measure algebra of a finite measure space. In turn, with each finite measure space there is associated an essentially unique measure space, having the same measure algebra, and consisting of a Stone space with a faithful normal measure. It is objects of this latter type—variously called Kakutani, perfect, or hyperstonian spaces ([1], [4])—which are studied in this work. In this selection of underlying spaces and in subsequent terminology, something of a middle course has been elected. On the one hand, many of the results carry over to general finite measure spaces; and on the other hand, the spaces actually studied are abelian W^*-algebras, as known representation theorems show. Neither fact is elaborated, the first because the really significant statements pertain to measure algebras and not to measure spaces, and the second, because use of operator terminology and methods would obscure the measure theoretic character of the study.

Let Γ be a compact hausdorff space, and let M denote the *-algebra $C(\Gamma)$ of complex-valued continuous function on Γ. If each bounded set f_a of real-valued functions in M has a least upper bound $f = \mathrm{LUB}\, f_a$ in M, then Γ is called a *Stone space* (see [10]). Following Dixmier [1], let Γ be a Stone space, and suppose that M admits a positive linear functional (or "measure," after Bourbaki) λ with these properties: 1) λ is *faithful*, that is, if $f \geqq 0$ and $\lambda(f) = 0$, then $f = 0$; 2) λ is *normal*, that is, for each bounded (non-void) lattice f_a of real-valued functions in M, $\lambda(\mathrm{LUB}\, f_a) = \mathrm{LUB}\, \lambda(f_a)$. [For Stone spaces, normality is equivalent to the requirement that compact sets with null interior have measure 0.] Then the space Γ is called *hyperstonian*, and the couple (Γ, λ) is called a *hyperstonian measure space*. Corrupting this usage slightly, we shall call a couple (M, λ) an *abstract hyperstonian measure space* if first, M is a commutative $(B^* = C^*)$-algebra with identity [that is,

a commutative Banach *-algebra with identity such that $\| AA^* \| = \| A \|^2$, for all A]; second, if relative to the customary order on M, positive elements being those of the form AA^*, each bounded (non-void) set of self-adjoint elements has a least upper bound; and finally, if λ is a positive faithful normal linear functional on M.

The key property of hyperstonian measure spaces is this: each bounded (Borel) measurable function f on Γ is equal almost everyhere (λ) to a continuous function ([1, Prop. 2]). In particular, the characteristic function of any measurable set is equal a.e. (λ) to a *projection*, that is, the characteristic function of a clopen set. Projections in $M = C(\Gamma)$ form a complete boolean algebra, and the measure algebra of (Γ, λ) can be identified with (M_P, λ), where M_P denotes the set of projections in M. This measure algebra (M_P, λ) completely determines (M, λ): if N is another abstract hyperstonian measure space, and if there exists a boolean isomorphism ϕ of M_P on N_P which conserves measure, then ϕ extends uniquely to a measure preserving (abbreviated, MP) isomorphism of M on N. An abstract hyperstonian measure space (M, λ) is called *non-atomic* if its measure algebra (M_P, λ) is non-atomic, that is, if each non-zero projection dominates projections of arbitrarily small positive measure.

If $(S, \mathcal{S}, m)$ is a general finite measure space, then, up to MP isomorphism, there exists precisely one abstract hyperstonian measure space (M, λ) with the same measure algebra as $(S, \mathcal{S}, m)$. One can construct this (M, λ) by taking for M the *-algebra of multiplications $L_f g = fg$ by bounded measurable functions f on $L_2(S, \mathcal{S}, m)$ under the operator norm, and for λ the linear functional $\lambda(L_f) = \int f(x) dm(x)$ on M. The M resulting in this construction is actually a commutative W^*-algebra of operators on a hilbert space, and λ of the construction has a representation $\lambda(A) = (Ax, x)$, for some vector x (e.g. the function 1) in that hilbert space. A theorem of Dixmier-Pallu de la Barriere asserts that any finite hyperstonian measure space has this form (up to MP isomorphisms) [1, Th. 2], so as noted above, these entities (M, λ) are in reality abelian W^*-algebras taken with faithful normal states.

Consider now an abstract non-atomic hyperstonian measure space (M, λ), with λ normalized by $\lambda(I) = 1$, fixed once and for all in the subsequent discussion of this section. A *-subalgebra N of M with identity is called hyperstonian if the couple (N, λ) is an abstract hyperstonian measure space in its own right. For this, it is sufficient that N contain, along with each bounded family A_a of self-adjoint elements, the elements $\mathrm{LUB}\, A_a$ of M.

(In fact, the validity of this condition implies that N is uniformly closed in M, and therefore B^*; the other conditions follow automatically.)

Let N be a hyperstonian subalgebra of M. Then for each A in M, there exist an element $E_N(A)$ of N such that

$$\lambda(AB) = \lambda(E_N(A)B), \text{ for all } B \text{ in } N. \tag{2.1}$$

This follows readily from application of the Radon-Nikodym theorem and the fact that bounded measurable function are equal a.e. to continuous functions on hyperstonian spaces (in this instance, the spectrum of N). We call $E_N(A)$ the *conditional expectation of A relative to N*. Because λ is faithful, it is clear that condition (2.1) uniquely determines $E_N(A)$. The mapping E_N has the following basic properties:

(2.2) $E_N(I) = I$,

(2.3) $A \rightarrow E_N(A)$ is a *-linear positive-definite mapping,

(2.4) $E_N(AB) = E_N(A)B$, for all $A \in M$, $B \in N$,

(2.5) $\text{LUB}\, E_N(A_a) = E_N(\text{LUB}\, A_a)$, for each bounded lattice A_a of self-adjoint elements.

Verifications follow trivially from (2.1) and the ascribed properties of λ. Conversely, if E_N is a mapping of M into itself satisfying (2.1)-(2.5), and if $N = \text{range}\, E_N$, then it follows readily that N is a hyperstonian subalgebra of M and E_N is the conditional expectation relative to N. Certain additional properties of E_N will enter. First, if α is an MP automorphism of M under which N is setwise invariant, then the relation

$$\alpha(E_N(A)) = E_N(\alpha(A)) \tag{2.6}$$

holds, this by the uniqueness of mappings E_N satisfying (2.1). Second, use of the inequality $\lambda[(A - E_N(A))(A - E_N(A))^*] \geqq 0$ establishes

$$\lambda(E_N(A)E_N(A^*)) \leqq \lambda(AA^*). \tag{2.7}$$

Finally, defining the *N-carrier* of an element A in M to be the greatest lower bound of projections P in N satisfying $PA = A$, one observes that, if $A \geqq 0$, then A and $E_N(A)$ have the same N-carrier.

A notion of type, related to the dimension type concept of operator theory, is introduced as follows. Let N be a hyperstonian subalgebra of M. A non-zero projection P in M is called *abelian* over N if each projection Q in M, $Q \leqq P$, has the form $Q = PC$, for some C in N_P. [One has, equivalently,

$Q \leqq P$ entails $E_N(Q) = CE_N(P)$, for some C in N.] We say that N is a *type I subalgebra* of M if each non-zero projection in M dominates a projection abelian over N. On the other hand, if M contains no projections abelian over N, we say that N is a *type II subalgebra* of M.

Let G be a group of MP automorphisms $A \rightarrow gA$ of M. [Throughout, by convention, automorphism = *-automorphism.] By the *fixed algebra* of G we mean the algebra $Z = [A \in M \mid gA = A$, for all $g \in G]$. Necessarily, Z is hyperstonian, and the preceding concepts of type apply. G is called type I (respectively, type II) group if Z is a type I (respectively, type II) subalgebra of M. Mixed types can occur, of course, but a natural device enables one to split G into purely type I and type II parts. Expressly, since the translate by any member of G of a type I projection over Z is again a type I projection over Z, it follows that the Z-carrier of a type I projection is again a type I projection over Z. In turn, the least upper bound C of all type I projection will be a type I projection which lies in Z, and the projection $I - C$ will be of type II. Now CM and $(I-C)M$ are hyperstonian algebras (with identities C and $I-C$) of types I and II, each reduces G, and M is the direct sum of CM and $(I-C)M$. When G is full (Section 3), G splits into the direct sum $G_C + G_{(I-C)}$ of two groups, the first a type I group of MP automorphisms of CM, the second a type II group MP automorphisms of $(I-C)M$. The summands are obviously uniquely determined. Thus the study of MP automorphism groups reduces to the study of groups which are either of type I or of type II.

A useful technical device for the type II theory can be adapted from a lemma of Maharam [6], used in her classification of homogeneous measure algebras.

MAHARAM'S LEMMA. *Assume that N is a type II subalgebra of M. Then under the expectation mapping E_N, the set of projections Q in M, which are dominated by some fixed P in M_P, maps onto the set of A in N with $0 \leqq A \leqq E_N(P)$.*

This is proved by an argument identical with Maharam's; the present condition, that P dominates no abelian projections, functions in the same way in proof as Maharam's condition that the cardinality of the principal ideal PM_P exceed that of PN_P, for each non-zero P. For the sake of completeness, we give a brief sketch of details. First one notes that, by normality of E_N, it suffices to show that, given $A \neq 0$ in N with $0 \leqq A \leqq E_N(P)$, there exists a non-zero projection $Q \leqq P$ such that $E_N(Q) \leqq A$. Now one makes a

further reduction. Let C be a projection in N with the property that $E_N(P)$ is positive on the clopen subset of the spectrum Γ_N of N corresponding to C, and let n be a positive integer. For proof, it suffices to show that a projection $Q \leqq P$ exists such that $0 < E_N(Q) \leqq E_N(P)/2^n$ on C. One arrives at such a Q as follows. The key step is to observe that, because P dominates no abelian projections, a projection R will exist such that $0 < E_N(R) < E_N(P)$ on C. Choosing such an R, let D_1 (respectively, D_2) be the projection in N corresponding to the sets

$$[\gamma \text{ in } \Gamma_N \mid 2E_N(R)(\gamma) \geqq E_N(R)(\gamma)], \qquad [\gamma \text{ in } \Gamma_N \mid 2E_N(R)(\gamma) \leqq E_N(P)(\gamma)];$$

the projection $Q_1 = D_1(P - R) + D_2R$ then satisfies $0 < E_N(Q_1) \leqq E_N(P)/2$ on C. Continuing this construction, with Q_1 replacing P, etc., one arrives at the desired Q, and the lemma follows.

Finally, we review certain action characteristics of measure preserving automorphisms. Let α be an MP automorphism of M. A projection P in M is said to be *absolutely fixed* under α if $\alpha(Q) = Q$, for each $Q \leqq P$. Plainly, there exists a maximal projection F_α absolutely fixed under α. We say that α is *freely-acting* if $F_\alpha = 0$. In turn, a group G of MP automorphisms of M is called freely-acting if $F_g = 0$, for each $g \neq e$ (the identity) in G. [It should be stressed that certain applications require a definition of free action which, for non-countable groups, is more stringent: specifically, under the latter definition, G is called freely acting if the set of points in the spectrum of M, each fixed under the induced action of some element $\neq e$ of G, is a set of measure zero.] This concept of free action (due to von Neumann) can be viewed as a natural relaxation of the notion of ergodicity, where by definition an automorphism is ergodic if it leaves no non-trivial projections fixed. Any ergodic automorphism is of course freely acting. However, the powers of α^n of an ergodic automorphism α need not be ergodic, whereas they are automatically freely acting in our non-atomic case. [If $\alpha^n(P) = P$, with α ergodic, then application of the ergodic theorem shows that $\lambda(P) = r/n$, for some integer $0 \leqq r \leqq n$, so plainly, α^n cannot leave arbitrarily small projections fixed.] As is customary, a group G of MP automorphisms is called *ergodic* if no non-trivial projections are simultaneously fixed under all members of G.

The construction of examples in this theory frequently involves representation of abstract groups as freely-acting automorphism groups. To assure existence of such representations, we append the following lemma.

LEMMA 2.1. *Any infinite discrete group can be faithfully represented*

as an ergodic group of freely-acting MP automorphisms of a non-atomic measure algebra.

Proof. Let S be a torsion-free discrete abelian group, and let α be an automorphism of S. The character group $\hat{S}$ of S is a connected compact abelian group, and α induces an automorphism $\hat{\alpha}$ of $\hat{S}$ defined by the relation $(\hat{\alpha}\hat{s}, s) = (\hat{s}, \alpha s)$, where $s \in S$ (respectively, $\hat{s} \in \hat{S}$) and $(\hat{s}, s)$ denotes the value of s in the character $\hat{s}$. Automatically, $\hat{\alpha}$ is haar measure preserving. In addition, we claim, $\hat{\alpha}$ is either freely-acting or trivial. Suppose there exists a measurable set E_1 in $\hat{S}$ with positive measure which is absolutely fixed under $\hat{\alpha}$, that is, for each measurable subset F of E_1, F and $\hat{\alpha}F$ differ by a set of measure 0. It follows easily that some measurable subset E of E_1 with positive measure will be pointwise fixed under $\hat{\alpha}$. In turn, $E^{-1}E$ is pointwise fixed. Because $E^{-1}E$ contains a neighborhood of the identity and $\hat{S}$ is connected, it follows that $\hat{\alpha}$ is the identity.

Applying this, let G be an infinite discrete group, and let S be the additive abelian group of integer-valued functions on G with compact support. Each g in G implements an automorphism $(\alpha_g s)(h) = s(gh)$ of this torsion-free group S, and it is clear that $g \to \hat{\alpha}_g$ is a faithful representation of G as a group of MP automorphisms of $\hat{S}$ under haar measure. The first paragraph shows that this action of G on $\hat{S}$ is free. We observe, finally, that these $\hat{\alpha}_g$ form an ergodic group. In fact, if a bounded measurable function $f \neq 0$ on $\hat{S}$ is invariant under all $\hat{\alpha}_g$, then the fourier transform $\hat{f}$ of f, as a function on S, will be constant on each orbit of G in S. Because f is square-integrable, and the orbit of each $s \neq 0$ in S under G is infinite, it follows that $\hat{f}$ must vanish except at 0, and therefore, that f is constant a. e. This proves the lemma, since the $\hat{\alpha}_g$ in particular determine automorphisms of the measure algebra of $\hat{S}$.

3. Full groups of measure preserving automorphisms. Throughout this section, (M, λ) will denote a fixed non-atomic abstract hyperstonian measure space.

Definition 3.1. Given two automorphisms α and β of M, denote by $F(\alpha, \beta)$ the maximal projection in M absolutely fixed under $\alpha^{-1}\beta$. [Clearly, $F(\alpha, \beta) = F(\beta, \alpha)$.] If G is any group of MP automorphisms of M, and if α is any automorphism of M, we say that α depends on G if $\mathrm{LUB}_{g \in G} F(\alpha, g) = I$ (the identity projection). Denote by $[G]$ the collection of all automorphisms of M which depend on G. $[G]$ is called the *full group* determined by G.

and a group G is called full if $G = [G]$. Two groups G_1 and G_2 of MP automorphisms of M are *equivalent* if they determine the same full group, that is, if $[G_1] = [G_2]$.

The condition that an automorphism α of M depend on a group G can be phrased in terms of the spectrum Γ of M: expressly, α lies in $[G]$ if and only if there exists a set Γ_0 in Γ having measure 0 and such that, for each γ in $\Gamma - \Gamma_0$, a clopen set P containing γ and a g in G exist such that the homeomorphisms of Γ induced by α and g agree at all points of P. (The exceptional set Γ_0 will be closed and nowhere dense.) The conventional notion of equivalence of groups G_1 and G_2 requires the existence of an MP automorphism φ of M such that $\varphi^{-1}G_1\varphi = G_2$. In particular, this implies that G_1 and G_2 are isomorphic as groups. Nothing of this sort is true for the present notion of equivalence, however. The one obvious necessary condition for equivalence, as the following lemma shows, is that G_1 and G_2 have the same fixed algebra; after this, particularly in the type *II* case, it becomes a highly technical problem to obtain revealing criteria for equivalence. This difficulty does not affect the theory, however, because full groups are the natural objects to study, and significant statements are invariant under equivalence.

LEMMA 3.1. *For any group of MP automorphisms of M, $[G]$ is again a group of MP automorphisms of M, and $[[G]] = [G]$. If G_1 is any subgroup of $[G]$, then the fixed algebra of G_1 contains that of G. Finally, elements α of $[G]$ are precisely those endomorphisms of M having a representation*

$$\alpha(P) = \sum_n P_n\beta_n(P), \tag{3.1}$$

where the $\beta_n \in G$, and P_n (resp. $\beta_n^{-1}(P_n)$) is a mutually orthogonal set of projections in M having least upper bound I.

Proof. First, any α in $[G]$ is automatically measure preserving. In fact, this follows because each $P \neq 0$ dominates a $Q \neq 0$ on which α is MP.

Next, we assert, α in $[[G]]$ entails α in $[G]$. For this, it will suffice to show that, given $P \neq 0$, there exists a $g \in G$ such that $F(\alpha, g)P \neq 0$. Choose β in $[G]$ such that $F(\alpha,\beta)P \neq 0$. Because $\mathrm{LUB}_g F(\beta, g)P = P$, we can choose $g \in G$ so that $F(\alpha,\beta)F(\beta,g)P \neq 0$. But $F(\alpha,\beta)F(\beta,g) \leqq F(\alpha,g)$, hence $F(\alpha,g)P \neq 0$.

To show that $[G]$ is a group, it suffices to prove that, given α and β in $[G]$ and $P \neq 0$, there exists an element $k \in G$ such that $F(\alpha^{-1}\beta, k)P \neq 0$. To do this, choose g in G such that $F(\beta, g)P \neq 0$. Because $\mathrm{LUB}_h\, \beta^{-1}\alpha(F(\alpha, h))$

$=I$, we can choose $h \in G$ to satisfy $\beta^{-1}\alpha[F(\alpha,h)]F(\beta,g)P \neq 0$. Consider any Q dominated by the latter projection. Since $\alpha^{-1}\beta Q \leqq F(\alpha,h)$, we have $h\alpha^{-1}\beta(Q) = \alpha(\alpha^{-1}\beta(Q)) = \beta(Q) = gQ$, so $\alpha^{-1}\beta Q = h^{-1}gQ$. Setting $k = h^{-1}g$, we have $F(\alpha^{-1}\beta, k)P \neq 0$, as desired.

If α lies in $[G]$, and if $gP = P$, for all g in G, then

$$\alpha P = \mathrm{LUB}_g\, \alpha[F(\alpha,g)P] = \mathrm{LUB}_g\, g[F(\alpha,g)P] = P.$$

In particular, therefore, P lies in the fixed algebra of any subgroup G_1 of $[G]$.

Fix an α in $[G]$. To show that α has a representation (3.1), we apply Zorn's lemma to construct a maximal set of mutually orthogonal non-zero projections P_a, subject to the requirement that each P_a be dominated by a projection $\alpha F(\alpha,\beta)$, for some β in G depending on P_a. Of necessity, P_a is countable set, and we write $P_n \leqq \alpha F(\alpha,\beta_n)$. The projection $P = I - \sum_n P_n$ must be 0; otherwise, because $\mathrm{LUB}_\beta\, \alpha F(\alpha,\beta) = I$, there will exist a non-zero projection of the form $P\alpha F(\alpha,\beta)$, contradicting maximality. Now $\alpha^{-1}P_n \leqq F(\alpha,\beta_n)$, hence $\alpha(\alpha^{-1}P_n) = \beta_n(\alpha^{-1}P_n)$, and $\beta_n^{-1}P_n = \alpha^{-1}P_n$. Therefore, $\sum \beta_n{}^{1}P_n = \sum \alpha^{-1}P_n = I$, $\sum$ denoting LUB. Finally, $\alpha(P) = \sum P_n\alpha(P) = \sum \alpha(P\alpha^{-1}P_n) = \sum \beta_n(P\beta_n^{-1}P_n) = \sum P_n\beta_nP$, proving (3.1). Conversely, if P_n (resp. $\beta_n^{-1}P_n$) is a set of mutually orthogonal projections with LUB I, and if we set $\alpha(P) = \sum P_n\beta_nP$, $\beta(P) = \sum \beta_n^{-1}(P_nP)$, then direct computation shows that both α and β are boolean endomorphisms of M_P and that $\alpha\beta P = \beta\alpha P = P$, for all P. [To justify this computation, one should bear in the relations $\mathrm{LUB}_{a,b}\, P_bQ_a = (\mathrm{LUB}\, P_a)(\mathrm{LUB}\, Q_b)$ and $\mathrm{LUB}\, P_a \cup \mathrm{LUB}\, Q_b = \mathrm{LUB}\, P_a \cup Q_a$, which hold for arbitrary sets P_a, Q_b in M_P, a and b describing a common index set.] Therefore, α and β are boolean isomorphisms of M_P and $\beta = \alpha^{-1}$. Both conserve measure: e. g.

$$\lambda(\alpha P) = \sum \lambda(P_n\beta_nP) = \sum \lambda(\beta_n^{-1}(P_n)P) = \lambda(P).$$

This completes the proof.

Consider a group G of MP automorphisms of M, with fixed algebra denoted Z. If $\alpha \in [G]$, and if $\alpha P \leqq Q$, for certain projections P and Q in M, then properties (2.3) and (2.6) of the mapping E_Z give $E_Z(Q) \geqq E_Z(\alpha P) = \alpha E_Z(P) = E_Z(P)$. A key fact is, the converse holds.

LEMMA 3.2. *Let G be a group of MP automorphisms of M with fixed algebra Z. If $E_Z(P) \leqq E_Z(Q)$, for projections P and Q in M, then there exists an α in $[G]$ with these properties: $\alpha P \leqq Q$; α^2 is the identity; and α is the identity off $P \cup \alpha P$.*

Proof. Assume, to begin, that $PQ = 0$. We have noted earlier that the Z-carrier $\bar{P} = \mathrm{LUB}_{\beta \in [G]} \beta P$ of P coincides with the Z-carrier (that is, the support) of $E_Z(P)$. Therefore, $E_Z(P) \leqq E_Z(Q)$ forces $\bar{P} \leqq \bar{Q}$. It follows that $[G]$ contains an α_1 such that $\alpha_1(P)Q \neq 0$. If we set $Q_1 = \alpha_1(P)Q$, $P_1 = \alpha_1^{-1}(Q_1)$, then (P_1, Q_1, α_1) is a triple with these properties: $\alpha_1 \in [G]$, $0 \neq P_1 \leqq P$, and $Q_1 = \alpha_1(P_1) \leqq Q$. By Zorn, we may construct a maximal set (P_a, Q_a, α_a) of triples satisfying these conditions, and subject to the further requirement that the P_a (respectively the Q_a) be a mutually orthogonal set. This maximal set will contain only countably many members (P_n, Q_n, α_n), and further, by the normality of E_Z, we have $E_Z(P - \sum P_n) \leqq E_Z(Q - \sum Q_n)$. Maximality forces $P = \sum P_n$: otherwise, the above argument applied to $P - \sum P_n$ and $Q - \sum Q_n$ would yield a new triple orthogonal to all the (P_n, Q_n, α_n). Let $\alpha = \alpha_n$ on P_n, α_n^{-1} on Q_n, and the identity off $P \cup \alpha P$. By construction and (3.1), α satisfies the conditions of the lemma. The general case, in which no restriction is made on PQ, follows directly if we apply the above case to $P - PQ$ and $Q - PQ$.

The following specialization of Lemma 3.2 is useful.

LEMMA 3.3. *Let G be a group of MP automorphisms of M, with fixed algebra Z, and let $P_0, P_1, \cdots, P_{n-1}$ be mutually orthogonal projections in M with $E_Z(P_0) = E_Z(P_1) = \cdots = E_Z(P_{n-1})$. Then $[G]$ contains an α with these properties: $\alpha P_i = P_{i+1}$ (indices* mod n), *α^n is the identity, and α is the identity off $P_0 + \cdots + P_{n-1}$.*

Proof. For each $i > 0$ choose β_i in $[G]$ such that $\beta_i(P_0) = P_i$, β_i^2 is the identity, and β_i is the identity off $P_0 + P_i$. Put $\alpha = \beta_{n-1} \cdots \beta_1$. Then if $Q \leqq P_i$, $\alpha Q = \beta_{i+1}\beta_i Q \leqq P_{i+1}$, $\alpha^2 Q = \beta_{i+2}\beta_i Q$, etc., until $\alpha^n Q = \beta_{i+n}\beta_i Q = \beta_i^2 Q = Q$.

LEMMA 3.4. *Let α be an arbitrary MP automorphism of M, and G a given group of MP automorphisms. Then there evists a unique maximal projection $E([G], \alpha)$ and a β in $[G]$ with the property that, for all Q in M, $E([G], \alpha)\alpha(Q) = E([G], \alpha)\beta(Q)$. One has*

$$E([G], \alpha) = \mathrm{LUB}_{\gamma \in [G]} \alpha F(\alpha, \gamma) = \alpha F(\alpha, \beta). \tag{3.2}$$

Proof. If no pair $P \neq 0$, $\beta \in [G]$ exists such that $P\alpha Q = P\beta Q$, for all Q, set $E([G], \alpha) = 0$. In the contrary case, denote by P_a, β_a the collection of all such pairs, and set $P = \mathrm{LUB}\, P_a$. Clearly, we can choose from P_a, β_a a subsequence P_n, β_n such that the P_n are mutually orthogonal and $P = \sum P_n$. By construction, $P_n \alpha Q = P_n \beta_n Q$, for all Q and n, whence

$$\alpha^{-1}(P_n)Q = \alpha^{-1}(P_n)\alpha^{-1}\beta_n(Q),$$

and substitution of $Q=\beta_n^{-1}(P_n)$ here gives $\alpha^{-1}(P_n)\leqq\beta_n^{-1}(P_n)$. This inequality reverses by symmetry, yielding $\alpha^{-1}(P_n)=\beta_n^{-1}(P_n)$. Using this, we have $E_Z(\alpha^{-1}P)=\sum E_Z(\alpha^{-1}P_n)=\sum E_Z(\beta_n^{-1}P_n)=\sum E_Z(P_n)=E_Z(P)$, where Z denotes the fixed algebra of G. Therefore, by Lemma 3.2, there exists a β_0 in $[G]$ such that $\beta_0(I-P)=\alpha^{-1}(I-P)$, $\beta_0{}^2=$ identity, $\beta_0=$ identity off $(I-P)\cup\alpha^{-1}(I-P)$. Setting $P_0=I-P$, we observe that P_n (resp. $\beta_n^{-1}P_n$) is for $n\geqq 0$ a mutually orthogonal set with LUB I. Applying Lemma 3.1, it follows that $\beta^{-1}(Q)=\sum_{n=0}^{\infty}\beta_n^{-1}(P_nQ)$ defines an element β of $[G]$. Now

$$P\beta Q=\beta(Q\beta^{-1}P)=\beta[\sum_{n-1}^{\infty}\beta_n^{-1}(P_n)Q]$$

$$=\beta[\sum_{n-1}^{\infty}\beta_n^{-1}(P_n\beta_n(Q))]=\beta[\sum_{n=1}^{\infty}\beta_n^{-1}(P_n\alpha Q)]=\beta[\beta^{-1}(P\alpha Q)]=P\alpha Q.$$

This projection $P=E([G],\alpha)$ has the ascribed properties, maximality following by construction. Turning to (3.2), one has, first, $\mathrm{LUB}_{\gamma\in[G]}\alpha F(\alpha,\gamma)\leqq P$; in fact, if γ in $[G]$ and $R\neq 0$ satisfy $R\leqq\alpha F(\alpha,\gamma)$ and $PR=0$, then R,γ is a pair with R orthogonal to all P_a, a contradiction. Next, $P\leqq\alpha F(\alpha,\beta)$: if $Q\leqq\alpha^{-1}P$, then $Q=(\alpha^{-1}P)Q=\alpha^{-1}P\alpha^{-1}\beta Q\leqq\alpha^{-1}\beta Q$, forcing $\alpha Q=\beta Q$, and $\alpha^{-1}P\leqq F(\alpha,\beta)$. The final link, $\alpha F(\alpha,\beta)\leqq\mathrm{LUB}_{\gamma\in[G]}\alpha F(\alpha,\gamma)$, is obvious.

Given a group G of MP automorphisms of M with fixed algebra Z, and a non-zero projection P in M, we introduce the local group (discussed in Section 1) by setting $[G]_P=[\alpha \text{ in } [G] \mid \alpha \text{ leaves } I-P \text{ absolutely fixed}]$.

LEMMA 3.5. *If Z' denotes the fixed algebra of $[G]_P$, then $Z'P=ZP$. In particular, if G is of type II, then so is the summand $[G]_PP$ of $[G]_P$.*

Proof. The inclusion $ZP\subset Z'P$ is clear. To prove equality, we proceed by indirect proof, assuming $ZP\subset Z'P$ properly. There will exist, then, an $R\leqq P$, R in Z', such that the Z-carriers of $P-R$ and R overlap, whence $\alpha(R)(P-R)\neq 0$, for some α in $[G]$. In particular, for some non-zero $R_1\leqq R$, $\alpha(R_1)=R_2\leqq P-R$. Set $\beta=\alpha$ on R_1, α^{-1} on R_2, identity elsewhere. Then β lies in $[G]_P$ and $\beta(R_1)=R_2$. Because β must leave R fixed, we have $R_2\beta R_1\leqq(P-R)\beta R=(P-R)R=0$, which is absurd, so the first statement of the lemma is proved. Now if $[G]_PP$ is not of type II, then there exists a projection $Q\leqq P$ abelian over Z'. But $Z'P=ZP$, so Q is abelian over Z, and G cannot be of type II. This proves the lemma.

PROPOSITION 3.1. *Let G be a full type II group of MP automorphisms of M, and let Z be the fixed algebra of G. Then* 1) *any G-invariant*

intermediate hyperstonian subalgebra N, $M \supset N \supset Z$, *has the form* $N = ZC + M(I - C)$, *for some* C *in* Z_P, *and* 2) *any full normal subgroup* K *of* G *has the form* $K = G_C$, *for some* C *in* Z_P.

Proof. Let C be the maximal projection in Z with the property $CZ = CN$. By dropping down to the summand $G_{(I-C)}$ on $(I - C)M$, we may assume $C = 0$. This done, we claim first that the N of assertion (1) is of type II over Z. Suppose to the contrary that a projection P in N is abelian over Z (relative to N). Because $C = 0$, P does not lie in Z, and we have

$$A = E_Z(P) \wedge E_Z(I - P) \neq 0.$$

Therefore, by Maharam's lemma, there exist projections P_0, P_1, P_2 in M with the following properties: P_0 and P_1 are dominated by P, $P_2 \leqq I - P$; the P_i are mutually orthogonal; and $E_Z(P_i) = A/2$, for each i. Applying Lemma 3.3, choose an α in G such that $\alpha(P_i) = P_{i+1}$ (indices mod 3), $\alpha^3 =$ identity, $\alpha =$ identity off $P_0 + P_1 + P_2$. Because N is G-invariant, the projection $P\alpha P$ lies in N, and further, because P is assumed abelian, $P\alpha P = PE$, for some E in Z. Now

$$\begin{aligned} P - P_0 = P\alpha P = PE = P\alpha(P\alpha P) &= P\alpha(P - P_0) \\ &= P\alpha P - P_1 = P - P_0 - P_1. \end{aligned}$$

This forces $P_1 = 0$, a contradiction, and it follows that N is of type II over Z. This being the case, let Q be any projection in M, and apply Maharam's lemma to conclude that $E_Z(P) = E_Z(Q)$, for some P in N. By Lemma 3.2, $\beta P = Q$, for some β in G. Therefore, $Q \in N$, by the G-invariance of N, and we have proved $N = M$.

Now let K be a full normal subgroup of G. Let C be the maximal projection in M left absolutely fixed by all β in K. For any α in G and any $Q \leqq \alpha C$, $\alpha\beta\alpha^{-1}Q = Q$, and $\alpha\beta\alpha^{-1} \in K$. Thus αC is also absolutely fixed under K, for all α in G. By maximality, therefore, $C \in Z$. As above, dropping to $G_{(I-C)}$ on $(I - C)M$ if necessary, we assume $C = 0$. This done, let Z_0 be the fixed algebra of K. Because K is normal, Z_0 is a G-invariant intermediate hyperstonian subalgebra of M, and so the first paragraph of proof together with the assumption $C = 0$ entail $Z_0 = Z$. By Maharam's lemma, choose a projection P such that $E_Z(P) = E_Z(I - P)$. By Lemma 3.2 applied to K, there exists a β in K such that $\beta P = I - P$, $\beta^2 =$ identity. Take any α in G_P. Then $\beta\alpha^{-1}\beta\alpha \in K$, and for $Q \leqq P$, $\beta\alpha^{-1}\beta\alpha Q = \beta^2\alpha Q = \alpha Q$. This proves that $K_P = G_P$. Likewise, $K_{(I-P)} = G_{(I-P)}$. Consider an arbitrary α in G. We

have $E_Z(P) = E_Z(\alpha P)$, so there exists a γ in K such that $\gamma^{-1}\alpha P = P$. Clearly, then, $\gamma^{-1}\alpha$ is the product of an element in $G_P = K_P$ and an element in $G_{(I-P)} = K_{(I-P)}$. This implies $\gamma^{-1}\alpha \in K$, so in turn, $\alpha \in K$. Therefore, $K = G$, and the proof is completed.

Proposition 3.1 holds without restriction to type *II* groups; the proof for the type *I* case follows easily out of results in the next section.

4. Groups of Type *I*. The structure of type *I* groups, as we shall see in this section, can be described rather completely: any type *I* group is "almost" equivalent to a finite group. Their technical interest lies in their very simplicity, in that one is led to see how much information can be gleaned about an arbitrary full group by study of its type *I* subgroups alone. Again, in this section, (M, λ) will denote an abstract non-atomic hyperstonian measure space.

LEMMA 4.1. *Let G be a group of MP automorphisms of M, let Z be the fixed algebra of G, and let Q be a projection abelian over Z. Then* 1) *if α and $\beta \in [G]$, and if for some projection P, $\alpha(P) \leqq Q$ and $\beta(P) \leqq Q$, then $\alpha(P) = \beta(P)$*; 2) *if $\alpha \in [G]$ is freely acting, then $\alpha(Q)Q = 0$*; 3) *if P is another projection abelian over Z with the same Z-carrier as Q, then $E_Z(P) = E_Z(Q)$.*

Proof. Re (1) Choose C and D in Z_P so that $\alpha(P) = PC$ and $\beta(P) = PD$. Then $\alpha(P - PC) = \alpha(P)(I - C) = 0$, so that $P = PC$. Likewise, $P = PD$. Therefore, $\alpha(P) = \alpha(PD) = QCD = \beta(PC) = \beta(P)$. Re (2) If $Q\alpha Q \neq 0$, then a non-zero $R \leqq Q$ exists such that $\alpha(R) \leqq Q$. Applying (1) with $\beta =$ identity, and P arbitrary $\leqq R$, it follows that $\alpha(P) = P$. Therefore, R is absolutely fixed under α, a contradiction. Re (3) Assume to the contrary that $E_Z(P) \neq E_Z(Q)$. We can assume that a $C \in Z_P$ exists such that $E_Z(QC) - E_Z(PC) \geqq 0$, but not $= 0$. By Lemma 3.2, an α in $[G]$ exists such that $\alpha(PC) \leqq QC$, $E_Z(QC - \alpha(PC)) \neq 0$. Hence there exists an $R \leqq QC$ such that $R\alpha(PC) = 0$. Denoting by $\bar{P}$ the Z-carrier of P, we have $R \leqq \bar{P}C$, so there exists a non-zero projection $S \leqq \bar{P}C$ and a $\beta \in [G]$ such that $\beta(S) \leqq R$. Now $\alpha(S) \leqq Q$, $\beta(S) \leqq Q$, and $\alpha(S)\beta(S) = 0$. This contradicts (1).

A corollary here is noteworthy. If G is infinite and freely acting, then (2) shows that no abelian projections can exist over the fixed algebra Z of G: were Q abelian over Z, then we would have $QgQ = 0$, for each g in G, $g \neq e$, and in turn, $gQhQ = 0$, for each pair $g \neq h$ in G; the set gQ is then infinite

and mutually orthogonal, contradicting the finiteness of λ. Therefore, *any freely acting infinite group is of type II.*

LEMMA 4.2. *Two type I MP groups G_1 and G_2 are equivalent if and only if they have the same fixed algebra.*

Proof. Necessity follows from Lemma 3.1. For sufficiency, denote by Z the common fixed algebra of G_1 and G_2, and consider some fixed α in $[G_1]$. If Q is any projection abelian over Z, and if β in $[G_2]$ is chosen so that $\beta(Q) = \alpha(Q)$, this being possible by Lemma 3.2, since $E_Z(Q) = E_Z(\alpha Q)$, then $Q \leqq F(\alpha, \beta)$; for any $P \leqq Q$ has the form $P = QC$ $(C \in Z)$, and hence $\alpha(P) = \alpha(Q)C = \beta(P)$. The identity I being the least upper bound of projections abelian over Z, it follows that $\mathrm{LUB}_{\beta \in [G_2]} F(\alpha, \beta) = I$, so $\alpha \in [G_2]$. Therefore, $[G_1] \subset [G_2]$, and equality follows by symmetry.

Definition 4.1. Let G be a group of MP automorphisms of M with fixed algebra Z. The group G is said to be of type I_n $(n = 1, 2, \cdots)$ if there exist mutually orthogonal projections $P_1, \cdots, P_n$ in M, each abelian over Z, such that $\sum_i P_i = I$ and $E_Z(P_1) = \cdots = E_Z(P_n)$. Such a set $P_1, \cdots, P_n$ is called an *abelian basis* for G.

LEMMA 4.3. (1) *Each type I group can be decomposed in one and only one way as a direct sum of type I_n groups* $(n = 1, 2, \cdots)$; (2) *Any group of type I_n is equivalent to a freely-acting cyclic group of order n, and any freely-acting group of order n is of type I_n*; (3) *A freely acting group G of order n has an abelian basis of the form $[gP \mid g \in G]$, and conversely, if P is any projection in M such that the gP are mutually orthogonal and have* LUB I, *then the gP form an abelian base for G.*

Proof. (1) We claim, first, that the identity I can be partitioned $I = \sum P_n$ as a sum of mutually orthogonal abelian projections in such a way that the Z-carriers C_i of P_i form a decreasing sequence; and that if $I = \sum P'_n$ is another partition of I with the same properties, then the Z-carrier of $P'_i = C_i$, for all i.

By definition, any non-zero projection Q dominates a projection P abelian over Z. This P can be chosen so that Z-carrier $P = Z$-carrier Q, as follows readily from the fact that, if R_n is a sequence of abelian projections with mutually orthogonal Z-carriers, then $\sum R_n$ is likewise abelian. To prove existence of the partition described above, let P_1 be an abelian projection with Z-carrier $C_1 = I$, and inductively, let P_n be an abelian projection

$\leqq I - \sum_{i<n} P_i$ with Z-carrier $C_n = Z$-carrier of $I - \sum_{i<n} P_i$. Plainly, $C_1 \geqq \cdots \geqq C_n \geqq \cdots$. Let C be the Z-carrier of $I - \sum P_n$. Then for all n, $C \leqq Z$-carrier $(I - \sum_{i<n} P_i) = C_n$, so the projections CP_n all have the same Z-carrier C. If $C \neq 0$, then (3) in Lemma 4.1 shows that $E_Z(P_1C) = E_Z(P_nC)$ for all n, whence $\lambda(I) \geqq \lambda[E_Z(P_1C + \cdots + P_nC)] = n\lambda(P_1C)$ for all n, contradicting the finiteness of λ. Let $I = \sum P'_i$ be another dissection of I into mutually orthogonal abelian projections with decreasing Z-carriers C'_i. Plainly $C'_1 = I = C_1$, and granting $C'_i = C_i$, for $i < n$, another application of (3) in Lemma 4.1 gives $E_Z(I - \sum_{i<n} P_i) = E_Z(I - \sum_{i<n} P'_i)$, and therefore,

$$C'_n = Z\text{-carrier}(\sum_{i\geqq n} P'_i) = Z\text{-carrier } (I - \sum_{i<n} P'_i) = Z\text{-carrier } (I - \sum_{i<n} P'_i)$$
$$= Z\text{-carrier } E_Z(I - \sum_{i<n} P_i) = Z\text{-carrier } (I - \sum_{i<n} P_i) = C_n.$$

This proves the assertion of the first paragraph.

Define $D_n = C_n - C_{n+1}$ $(n \geqq 1)$. Now $\sum_{k\leqq n} D_k = I - C_n$, and since either $C_n = 0$ for some n, or $C_n \neq 0$ for all n but GLB $C_n = 0$, as the above argument shows, we must have $\sum D_n = I$. Further, $D_n = D_nP_1 + \cdots + D_nP_n$, the projections D_nP_i are abelian with Z-carrier D_n, and therefore, $E_Z(D_nP_1) = \cdots = E_Z(D_nP_n)$. It follows that $[G] = \sum [G]_{D_n} D_n$, and $[G]_{D_n}$ on D_nM is by definition of type I_n. Let D'_n be another sequence of mutually orthogonal projections in Z such that $\sum D'_n = I$, and $D'_n = R_1^{(n)} + \cdots + R_n^{(n)}$, with $R_i^{(n)}$ abelian and $E_Z(R_1^{(n)}) = \cdots = E_Z(R_n^{(n)})$. We will show that $D'_n = D_n$ for all n. In fact, let $P'_n = \sum_{k,k\geqq n} R_n^{(k)}$. The P'_i are abelian and $\sum P'_n = I$. Further, Z-carrier $P'_n = C'_n = \text{LUB}_{k,k\geqq n}$ Z-carrier $R_n^{(k)} = \sum_{k\geqq n} D'_k$. The C'_n consequently decrease with n, and the uniqueness assertion of the first paragraph implies $C'_n = C_n$, for all n. Hence, $D_n = C_n - C_{n+1} = C'_n - C'_{n+1} = D'_n$, and (1) is established.

(2) and (3). Let K be a group of type I_n, with fixed algebra denoted Z. Let $P_0, \cdots, P_{n-1}$ be an abelian base for K over Z. Because the P_i have the same expectation relative to Z, we can apply Lemma 3.3 to select an α in $[K]$ such that $\alpha P_i = P_{i+1}$ (indices mod n) and $\alpha^n =$ identity. Trivially, the group G generated by α is freely acting of order n, and by construction, $[G] \subset [K]$. To prove that $[G] = [K]$, it suffices by Lemma 4.2 to show that each P in the fixed algebra of $[G]$ already lies in Z. For this, let C be the projection in Z such that $PP_0 = CP_0$, and compute

$$P - C = (P - C)(\sum_{i=0}^{n-1} \alpha^i P_0) = (\sum_{i=0}^{n-1} \alpha^i [(P - C)P_0] = 0.$$

Let G be a freely acting group of order n. If g is any element of G not e, then by free action, given any non-zero projection Q, there exists a non-zero $R \leqq Q$ such that $RgR = 0$. Enumerate the elements of G not e as $g_1, \cdots, g_{n-1}$, let R_1 be a non-zero projection such that $R_1 g_1 R_1 = 0$, let R_2 be a non-zero projection $\leqq R_1$ such that $R_2 g_2 R_2 = 0$, etc. We arrive at a non-zero projection P $(= R_{n-1})$ such that $PgP = 0$, for all $g \neq e$. If $g \neq h$, then $gPhP = g[Pg^{-1}hP] = 0$. By Zorn, we can assume that no larger projection $P_1 \geqq P$ has this property $P_1 g P_1 = 0$ (all $g \neq e$). But then $\sum gP = I$: for $C = \sum gP$ lies in the fixed algebra Z of G, and were $I - C \neq 0$, we could apply the above argument to $I - C$ to obtain a projection P' orthogonalized by G and $\leqq I - C$; the projection $P' + P$ is then orthogonalized by G and is larger than P, a contradiction. The projection P is abelian over Z, since each $R \leqq P$ has the form $R = (\sum gR)P$ and $\sum gR \in Z$. Consequently, all the projections gP are abelian over Z, and therefore, G has an abelian base of the form prescribed in (3). This base has n elements, and therefore G is a type I_n. The lemma follows.

Let G be a type I group with fixed algebra Z. We will call R a *bounded* (type I) group if the identity is a finite union of projections abelian over Z. This means that G has only finitely many "pure," or type I_n, constituents, and a simple application of Lemma 4.3 (2) shows in fact that G is equivalent to a finite group. Conversely, it is evident that any group equivalent to a finite group must be a bounded type I. A related observation, useful in later work, is the following:

LEMMA 4.4. *Let F be a finite set of MP automorphisms of M which leave fixed (pointwise) a bounded type I subalgebra of M. Then the group G_F of MP automorphisms generated by F is of finite order.*

Proof. By hypothesis, $[G_F]$ is a finite direct sum of type I_n's. If we know that the restriction of G_F to each summand is of finite order, then the same will follow for G_F. So it will suffice to assume that G_F is of type I_n. By Lemma 4.3, there exists a freely acting finite group K of order n such that $[K] = [G_F]$. We can assume that $F = F^{-1}$ and that the identity lies in F, so that G_F is the multiplicative semigroup generated by F, By Lemma 3.1, we may represent each α in F in the form $\alpha = \sum_k Q_k^{(\alpha)} k$. Let $\mathfrak{B}$ be the smallest K-invariant boolean algebra of projections containing all the $Q_k^{(\alpha)}$. $\mathfrak{B}$ is obviously finite, as is the collection of all linear endomorphisms of M of the form $\sum_k R_k k$ $(R_k \in \mathfrak{B})$. But all products of the α in F lie in this collection. Therefore, G_F is finite.

We conclude with a result on the imbedding of bounded type I's in type I_n's.

LEMMA 4.5. *Let K be a bounded type I subgroup of a type II group G. Then for an appropriate s, there exists a type I_s subgroup L of $[G]$ such that $K \subset [L]$.*

Proof. To begin, suppose that K is a type I_n subgroup of $[G]$, and that $m = nr$ is an integer divisible by n. We will construct a type I_m subgroup L of $[G]$ such that $K \subset [L]$. Replacing K by an equivalent group if necessary, we can assume that K is freely acting. Choose a projection P such that the kP $(k \in K)$ form an abelian basis for K. Denote by Z the fixed algebra of G, and applying Maharam's lemma, partition $P = P_0 + \cdots + P_{r-1}$ as a sum of mutually orthogonal projections with $E_Z(P_i) = 1/m$. By Lemma 3.3, choose an α in $[G]$ such that $\alpha(P_i) = P_{i+1}$ (indices mod r), $\alpha^r =$ identity, $\alpha =$ identity off P. Next, define an automorphism β in $[G]$ as follows: if $Q \leqq kP$, then $\beta Q = k\alpha k^{-1}Q$. Let L be the group generated by K and β. It is easy to see that β commutes with each element of K, so elements of L have the form $k\beta^i$. Further, $\sum_k \sum_{i=0}^{r-1} k\beta^i P_0 = I$. Now L is freely acting: if not, then some $k_0\beta^{i_0} \neq e$ will certainly leave fixed a non-zero projection of the form $k_1\beta^{i_1}R$ $(R \leqq P_0)$. This gives $k_0k_1\beta^{i_0+i_1}R = k_1\beta^{i_1}R$. The left side is $\leqq k_0k_1P$, the right $\leqq k_1P$, and hence $k_0 = e$, and $\beta^{i_0+i_1}R = k_1\beta^{i_1}R$. This gives $\alpha^{i_0}R$, implying $i_0 = 0 \pmod r$ and $k_0\beta^{i_0} =$ identity, a contradiction. Therefore, L is a freely acting group of type I_m. Any projection in the fixed algebra of L has the form $\sum_k \sum_{i=0}^{r-1} k\beta^i R$, for $R \leqq P_0$, and such a projection is obviously invariant under K. This implies that $K \subset [L]$, and the first assertion is established.

In the general case, there will exist mutually orthogonal projections $C_1, \cdots, C_t$ in the fixed algebra of K such that $I = \sum C_i$ and $[K]_{C_i}$ on C_iM is of type I_{n_i}. Let $s =$ least common multiple of the n_i. Because $[G]_{C_i}$ is of type II (Lemma 3.5), we can apply the above construction to each $[K]_{C_i}$ on C_iM, imbedding these in type I_s subgroups L_i of $[G]_{C_i}$ on C_iM. Now, $L = \sum L_i$ is a type I_s subgroup of $[G]$ and $K \subset [L]$. This proves the lemma.

5. Approximately finite groups. We prepare now to confront our main problem, approximation of arbitrary MP automorphisms in type I subgroups. The initial step concerns a relaxation of conditions in Lemma 4.2. According to that lemma, if K is a type I group, then any MP automorphism

Z which leaves the fixed algebra Z_K of K pointwise fixed is already in $[K]$. This is generalized here to a statement that, if an MP automorphism α leaves Z_K "almost pointwise fixed," then α is "almost in $[K]$."

A numerical lemma, useful in several other connections, will provide the decisive estimate.

LEMMA 5.1. *Let (a_{ij}) be an $n \times n$ matrix with real entries. Assume that for each subset S of $\Lambda = (1, \cdots, n)$*

$$|\sum_{i \in S,\ j \in \Lambda - S} a_{ij}| < \epsilon. \tag{5.1}$$

Then $|\sum_{i \neq j} a_{ij}| < 4\epsilon$.

Proof. We can assume that n is a power of 2. For choose m such that $2^m > n$, and set $a_{ij} = 0$ whenever $n+1 \leqq i \leqq 2^m$ or $n+1 \leqq j \leqq 2^m$. The inequality (5.1) holds for the enlarged matrix, and the sum of off-diagonal terms remains unchanged. In what follows, then, we take $\Lambda = (1, \cdots, 2^m)$.

Partition Λ into two disjoint subsets Λ_0, Λ_1, each with 2^{m-1} elements, in such a way that the sum $s(0,1) = \sum_{i \in \Lambda_0,\ j \in \Lambda_1} (a_{ij} + a_{ji})$ is a maximum among corresponding sums for all such partitions of Λ. Denote by Δ_r the set of all r-tuples $(\epsilon_1 \cdots \epsilon_r)$, where $\epsilon_i = 0$ or 1. If the partition $\Lambda = \bigcup_{\delta \in \Delta_r} \Lambda_\delta$ is defined for $r < m$, with $|\Lambda_\delta| = 2^{m-r}$, we partition each Λ_δ into disjoint subsets $\Lambda_{\delta 0}$, $\Lambda_{\delta 1}$, each with 2^{m-r-1} elements, in such a way that

$$s(\delta 0, \delta 1) = \sum_{i \in \Lambda_{\delta 0},\ j \in \Lambda_{\delta 1}} (a_{ij} + a_{ji}) \tag{5.2}$$

is a maximum among corresponding sums for all such partitions of Λ_δ. Let $A_{r+1} = \sum_{\delta \in \Delta_r} s(\delta 0, \delta 1)$. Then

$$\sum_{i \neq j} a_{ij} = \sum_{r=1}^{m} A_r \text{ and } A_1 < 2\epsilon. \tag{5.3}$$

Now, we assert, $A_r \geqq 2A_{r+1}$, for each $r < m$. To see this, we apply the maximum property of the sums $s(\delta, \delta') = s(\delta', \delta)$ to compute

$$\begin{aligned}
2A_r &= 2 \sum_{\delta \in \Delta_{r-1}} s(\delta 0, \delta 1) \\
&= 2 \sum_{\delta \in \Delta_{r-1}} [s(\delta 00, \delta 10) + s(\delta 00, \delta 11) + s(\delta 01, \delta 10) + s(\delta 01, \delta 11)] \\
&= \sum_{\delta \in \Delta_{r-1}} [s(\delta 00, \delta 01) + s(\delta 00, \delta 10) + s(\delta 11, \delta 01) + s(\delta 11, \delta 10)] \\
&= \sum_{\delta \in \Delta_{r-1}} [s(\delta 00, \delta 01) + s(\delta 00, \delta 11) + s(\delta 10, \delta 01) + s(\delta 10, \delta 11)] \\
&= 2 \sum_{\delta \in \Delta_{r-1}} [s(\delta 00, \delta 01) + s(\delta 10, \delta 11)] + A_r = 2A_{r+1} + A_r,
\end{aligned}$$

establishing the inequality. Therefore,

$$A_1 \geqq 2A_2 \geqq \cdots \geqq 2^{m-1}A_m, \text{ and } \sum_{r=1}^{m} A_r \leqq \sum_{r=1}^{m} A_1/2^{r-1} < 2A_1 < 4\epsilon.$$

Application of the same argument to the matrix $(-a_{ij})$ completes the proof.

LEMMA 5.2. *Let K be a type I group with fixed algebra denoted Z, and let α be an MP automorphism of M such that $F(\alpha,\beta)=0$, for all $\beta\in K$. Then*

$$\sup_{C\in Z_P}\lambda(C\Delta\alpha C) \geqq \tfrac{1}{2}. \tag{5.4}$$

Proof. (Recall, throughout, that the measure λ is normalized by $\lambda(I)=1$. The Δ in (5.4) of course denotes symmetric difference.) The proof depends on the following fact:

(5.5) given $C\neq 0$ in Z_P, there exists a non-zero D in Z_P, $D\leqq C$, such that $D\alpha D=0$.

Assertion (5.5) will be established by an indirect argument: assume some non-zero C_0 in Z_P has the property that $D\alpha D\neq 0$, for each non-zero $D\leqq C_0$, D in Z. Using type I theory, we choose an F in Z_P such that $FC\neq 0$ and $[K]_F$ on FM is of type I_n, for an appropriate n. Write $F=P_1+\cdots+P_n$, where the P_i are abelian over Z and have the same expectation relative to Z.

Assume that for each i and each non-zero $E\leqq C_0F$, $E\in Z$, there exists a non-zero C in Z, $C\leqq E$, such that $C\alpha(C)P_i=0$. In particular then we can choose a non-zero $C_1\leqq C_0F$ such that $C_1\alpha(C_1)P_1=0$, a non-zero, $C_2\leqq C_1$ such that $C_2\alpha(C_2)P_2=0$, etc., arriving finally at a non-zero $C_n\leqq C_0F$ such that $C_n\alpha(C_n)P_i=0$, for all i. This gives

$$C_n\alpha(C_n)=C_n\alpha(C_n)F=\sum_i C_n\alpha(C_n)P_i=0.$$

This contradicts the basic assumption in our indirect proof. Therefore, there exists an i, which we take as 1, and a non-zero E in Z, $E\leqq C_0F$, such that $C\alpha(C)P_1\neq 0$, for all non-zero $C\leqq E$. We will have $\alpha(C)P_1\geqq CP_1$, for all $C\leqq E$; otherwise, for some such C, $CP_1(I-\alpha(C)P_1)\neq 0$, and if we choose D in Z_P so that $\alpha(C)P_1=DP_1$, then $C(I-D)P_1=CP_1(I-\alpha(C)P_1)$, whence $C(I-D)\neq 0$, because C is dominated by the carrier F of P_1; now on the one hand, $[C(I-D)]\alpha[C(I-D)]P_1\neq 0$, by choice of the pair (E,P_1), but on the other hand, $[C(I-D)]\alpha[C(I-D)]P_1\leqq[C(I-D)]\alpha(C)P_1 =C(I-D)DP_1=0$, a contradiction. Further, if we choose projections D_i

in Z, $D_i \leqq F$, such that $\alpha(P_i)P_1 = D_iP_1$, then for some i, $D_iE \neq 0$, for otherwise $E\alpha(F)P_1 = 0$, whence $E\alpha(E)P_1 = 0$, again a contradiction.

Now consider any C in Z, $C \leqq D_iE \neq 0$. Then $\alpha(CP_i)P_1 = \alpha(C)D_iP_1 \geqq CD_iP_1 = CP_1$. Therefore $\alpha(CP_i) \geqq CP_1$, and equality must hold because these projection have the same measure. It follows that $\alpha(CP_i) = CP_1$, for all $C \leqq D_iE \neq 0$. Because the abelian projections P_1 and P_i have the same expectation relative to Z, there exists a β in $[K]$ such that $\beta(P_i) = P_1$. Also $\beta(CP_i) = CP_1$, for all $C \leqq D_iE$, C in Z, and this shows that $D_iEP_i \leqq F(\alpha, \beta)$. But $D_iEP_i \neq 0$, because P_i has Z-carrier $F \geqq D_iE$. Therefore $F(\alpha, \beta) \neq 0$, for some β in $[K]$, hence for some β' in K, a contradiction, and (5.5) is established.

Turning to the lemma itself, we apply (5.5) to express I as a (necessarily countable) sum of mutually orthogonal non-zero projections D_i in Z satisfying $D_i\alpha D_i = 0$, for each i. Fix a $\delta > 0$. Choose n so that $\sum_{k=n}^{\infty} \lambda(D_k) < \delta$, and define $C_i = D_i$ $(i < n)$, $C_n = \sum_{i \geqq n} D_i$. Let

$$k = \sup_{C \in Z_P} \lambda(C \Delta \alpha C) = \sup_{C \in Z_P} \lambda[\alpha(C)(I - C) + C\alpha(I - C)].$$

Let $a_{ij} = \lambda[\alpha(C_i)C_j + C_i\alpha(C_j)]$, for $1 \leqq i, j \leqq n$. Then if $\Lambda = (1, \cdots, n)$, and $C = \sum_{i \in S} C_i$, we have $\sum_{i \in S,\ j \in \Lambda - S} a_{ij} = \lambda[\alpha(C)(I - C) + C\alpha(I - C)] \leqq k$. By Lemma 5.1, $\sum_{i \neq j} a_{ij} \leqq 4k$. But

$$\sum_i a_{ii} = \sum_i 2\lambda(\alpha(C_i)C_i) = 2\lambda(\alpha(C_n)C_n) < 2\delta,$$

and

$$\sum_{i,j} a_{ij} = 2\lambda(I) = 2 = \sum_i a_{ii} + \sum_{i \neq j} a_{ij} \leqq 2\delta + 4k.$$

Therefore, $k \geqq (1 - \delta)/2$, and since δ is arbitrary, $k \geqq \frac{1}{2}$.

LEMMA 5.3. *Let K be a type I group with fixed algeba Z, and let α be an arbitrary MP automorphism of M. Assume that* $\sup_{C \in Z_P} \lambda(C \Delta \alpha C) < \epsilon$. *Then*

$$\lambda(E([K], \alpha)) > 1 - 2\epsilon. \tag{5.6}$$

Proof. Write $E = E([K], \alpha)$. By definition, there exists a β in $[K]$ such that $E\alpha(P) = E\beta(P)$, for all P in M. Let $\tau = \alpha\beta^{-1}$. Then $\tau E = E$, τ leaves E absolutely fixed, and $F(\tau, \gamma) \leqq E$, for all $\gamma \in [K]$: in fact, $E\alpha P = E\beta P$ for all P implies $E\tau P = EP$ for all P, so in particular $P \leqq E$ gives $P = E\tau P \leqq \tau P$, forcing $P = \tau P$; and applying Lemma 3.4, we have for any γ in $[K]$, $F(\tau, \gamma) = F(\gamma, \alpha\beta^{-1}) = \beta F(\gamma\beta, \alpha) \leqq \beta\alpha^{-1}E = \tau^{-1}E = E$. Con-

sider the restricted group $[K]_{(I-E)}$ on $(I-E)M$. If we denote by Z_0 the fixed algebra of $[K]_{(I-E)}$ on $(I-E)M$, then Lemma 5.2 gives

$$\sup_{C \in Z_0} \lambda(C \Delta \tau C) \geqq \lambda(I-E)/2.$$

Now by Lemma 3.5, $Z_0 = Z(I-E)$. Also, for any C in Z, $\alpha C = \tau C = \tau(C(I-E)) + CE$. Therefore,

$$\lambda(I-E)/2 \leqq \sup_{C \in Z_0} \lambda(C \Delta \tau C) = \sup_{C \in Z} \lambda(C \Delta \tau C) = \sup_{C \in Z} \lambda(C \Delta \alpha C) < \epsilon,$$

so $1 - \lambda(E) < 2\epsilon$, and $\lambda(E[K], \alpha)) = \lambda(E) > 1 - 2\epsilon$.

Application of Lemma 5.3 in the case $K =$ identity yields a useful result of Halmos [3] on the equivalence of two natural metrics on the group of all MP automorphisms, which we discuss now in the current notation. The standard metric on the group of all MP automorphisms of M is defined by

$$d(\alpha, \beta) = \sup_{P \in M_P} \lambda(\alpha P \Delta \beta P). \tag{5.7}$$

Endowed with d, this group is a complete metric space, in which the natural notion of distance from an automorphism α to a full subgroup $[G]$ is given by

$$d(\alpha, [G]) = \inf_{\gamma \in [G]} d(\alpha, \gamma). \tag{5.8}$$

On the other hand, an apparently more revealing metrization of the group of all measure preserving automorphism is available. Expressly, define

$$\delta(\alpha, \beta) = \lambda(I - F(\alpha, \beta)). \tag{5.9}$$

It is easy to see that δ is a metric, the triangle inequality following from the relation $F(\alpha, \beta)F(\beta, \gamma) \leqq F(\alpha, \gamma)$. Like d, the metric δ is invariant: $\delta(\alpha, \beta) = \delta(\alpha\gamma, \beta\gamma) = \delta(\gamma\alpha, \gamma\beta)$, for all MP automorphism α, β, γ, this because $F(\alpha, \beta) = F(\gamma\alpha, \gamma\beta) = \gamma F(\alpha\gamma, \beta\gamma)$. For any MP group G, we have $\inf_{\gamma \in [G]} \delta(\alpha, \gamma) = 1 - \sup_{\gamma \in [G]} \lambda(F(\alpha, \gamma)) = 1 - \sup_{\gamma \in [G]} \lambda(\alpha F(\alpha, \gamma))$, and by formula (3.2) this becomes

$$\delta(\alpha, [G]) = \lambda(I - E([G], \alpha)). \tag{5.10}$$

As noted by Halmos, the key fact is that these metrics can be used interchangeably:

Lemma 5.4. *The metrics d and δ are equivalent, and any full group is complete in either metric.*

Proof. First we note that d is stronger than δ:

$$\begin{aligned}\lambda(\alpha P \Delta \beta P) &= \lambda(\alpha[P(I - F(\alpha, \beta))] \Delta \beta[P(I - F(\alpha, \beta))]) \\ &= \lambda[(P \Delta \alpha^{-1}\beta P)(I - F(e, \alpha^{-1}\beta)] \leqq \delta(\alpha, \beta),\end{aligned}$$

so that $d(\alpha,\beta) \leqq \delta(\alpha,\beta)$. On the other hand, applying Lemma 5.3 when K consists of the identity alone, and therefore $Z = M$ and $E([K],\alpha) = F(e,\alpha)$, we see that $d(e,\alpha) < \epsilon$ implies $\lambda(I - F(e,\alpha)) < 2\epsilon$, so that $\delta(e,\alpha) < 2\epsilon$. This and the invariance of the metrics d and δ now show that the identity map from any (G,d) to (G,δ) is uniformly continuous, proving that the metrics are equivalent. If α_n is a δ-cauchy sequence in a full group $[G]$, then α_n is automatically d-cauchy, and therefore converges in d-metric to some MP automorphism α. By equivalence, this convergence carries over to the δ-metric. But then, α lies in $[G]$, for $\lim_n \lambda(F(\alpha,\alpha_n)) = 1$ clearly entails $\mathrm{LUB}_{\gamma \in [G]} F(\alpha,\gamma) = I$, that is, $\alpha \in [G]$. Again by equivalence, $[G]$ is also complete in d.

[The fact that δ is stronger than d can of course be proved rather easily without benefit of Lemma 5.3. One procedure is this: let $\tau = \alpha^{-1}\beta$, $F = F(e,\tau)$, and choose a P maximal with the property that $P\tau P = 0$. Then $I = F + [\tau^{-1}P \cup (P + \tau P)]$, and

$$\delta(\alpha,\beta) = \lambda(I - F) \leqq \tfrac{3}{2}\lambda(P \Delta \tau P) \leqq \tfrac{3}{2} d(\alpha,\beta).]$$

PROPOSITION 5.1. *The following conditions on a group G of MP automorphisms are equivalent: for each finite set $\beta_1, \cdots, \beta_n$ in G and each $\epsilon > 0$,*

(1) *there exists a type I subgroup K of $[G]$ such that $\delta(\beta_j, [K]) < \epsilon$, for $i = 1, \cdots, n$;*

(2) *there exists a finite subgroup K of $[G]$ and elements $\beta'_1, \cdots, \beta'_n$ of K such that $d(\beta_i, \beta'_i) < \epsilon$, for $i = 1, \cdots, n$;*

(3) *there exists a type I subgroup K of $[G]$ such that*

$$\sup_{C \in Z_K} \lambda(C \Delta \beta_i C) < \epsilon,$$

for $i = 1, \cdots, n$, where Z_K denotes the fixed algebra of K.

Further, the validity of these conditions for G entails their validity for $[G]$, and therefore, for any group equivalent to G.

Proof. (1) implies (2). Trivially, one can assume that the K in condition (1) is of bounded type I. This done, let β'_i be an element of K such that $\delta(\beta_i, \beta'_i) < \epsilon$, so $d(\beta_i, \beta'_i) < \epsilon$. By Lemma 4.4, the group generated by the β'_i is finite, and (2) follows.

(2) implies (3). Let C lie in the fixed algebra of the group K of condition (2). Then

$$\lambda(\beta_i C\Delta C) \leqq \lambda(\beta_i C\Delta\beta'_i C) + \lambda(\beta'_i C\Delta C) = \lambda(\beta_i C\Delta\beta'_i C)$$
$$\leqq \sup_{P\in M_P} \lambda(\beta_i P\Delta\beta'_i P) = d(\beta_i, \beta'_i) < \epsilon.$$

(3) implies (1). This follows immediately from Lemma 5.3.

Turning to the last statement of the proposition, we assume condition (1) holds for elements of G, and prove that it will then hold for arbitrary $\beta_1, \cdots, \beta_n$ in $[G]$. For this, we represent $\beta_i = \sum_{k=1}^{\infty} Q_k^{(i)}\alpha_k^{(i)}$, by formula (3.1), where the $\alpha_k^{(i)} \in G$ and the sets $Q_k^{(i)}$ satisfy the orthogonality conditions of Lemma 3.1. Fix a $\delta > 0$. Choose an integer r such that $\lambda(\sum_{k\leqq r} Q_k^{(i)}) > 1 - \delta/2$, for all i. Applying condition (1), choose a type I subgroup K of $[G]$ such that $\delta(\beta, [K]) < \delta/2mn$, for each α in the set

$$F = [\alpha_k^{(i)} \mid 1 \leqq i \leqq n, 1 \leqq k \leqq r].$$

Define $E = \prod_{\alpha\in F} E([K], \alpha)$. It is easy to see that $\lambda(E) \geqq 1 - \delta/2$. For each α in F, denote by $\gamma(\alpha)$ the element of $[K]$ such that $E([K], \alpha)\alpha = E([K], \alpha)\gamma(\alpha)$. Then $E\alpha = E\gamma(\alpha)$, for all α, so $E(\sum_{k=1}^{r} Q_k^{(i)}\alpha_k^{(i)}) = \sum_{k=1}^{r} Q_k^{(i)}E\gamma(\alpha_k^{(i)})$ for all i. Lemma 3.4 now shows that $E([K], \beta_i) \geqq \sum_{k=1}^{r} Q_k^{(i)}E$. Therefore, $\lambda(E[K], \beta_i)) \geqq \lambda((\sum_{k=1}^{r} Q_k^{(i)})E) \geqq 1 - \delta$. This completes the proof.

Definition 5.1. The group G is called *approximately finite* if it satisfies any one of the equivalent condition (1)-(3) of Proposition 5.1.

Automatically, any type I group is approximately finite. Interest centers, of course, on approximately finite groups of type II. An example of an approximately finite group of type II—which serves as a prototype in subsequent developments—can be obtained as follows. Let A be an infinite index set, and for each $a \in A$, let G_a be an abstract group of order 2. Form the restricted direct product $\prod_a G_a = G$ of the G_a (where it is understood that all but a finite number of components of each g in G are identities). Lemmas 2.1 assures us that G has a faithful representation as a freely-acting group of MP automorphisms of an abstract non-atomic hyperstonian measure space. In this representation, G will automatically be approximately finite, for any finite subset of G generates a finite, and therefore type I, subgroup of G. Another class of examples of more classical interest is developed by

THEOREM 1. *Any singly-generated group of MP automorphism is approximately finite.*

Proof. Let α be an MP automorphism of M, and G the group of MP automorphisms generated by α. The following assertion affords the basis of proof:

(5.11) Let E be a non-zero projection which dominates no α-fixed projections. Then there exists a β in $[G]$ such that $\beta = \alpha$ on E, $\beta =$ identity off $E \cup \alpha E$, and β generates a type I subgroup of $[G]$.

First, we partition E into a possibly terminating sum $E = \sum P_n$ of mutually orthogonal projections P_n which satisfy $\alpha P_1 = (\alpha E)(I - E)$ and $\alpha P_{n+1} \leqq P_n$, for $n \geqq 1$. To do this, define $P_1 = E\alpha^{-1}(I - E)$, $P_0 = \alpha P_1$, and for $n \geqq 1$, $P_{n+1} = [E - (P_1 + \cdot\cdot\cdot + P_n)]\alpha^{-1}(P_1 + \cdot\cdot\cdot + P_n)$. By construction, the P_n are mutually orthogonal. Now if $R \leqq P_{n+1}$ and $\alpha R \leqq P_i$ $(0 < i \leqq n)$, then $R \leqq (\alpha^{-1}P_i)(E - (P_1 + \cdot\cdot\cdot + P_i)) \leqq P_{i+1}$, forcing $i = n$. Therefore $\alpha P_{n+1} \leqq P_n$. If $P_{n+1} = 0$, then $E - (P_1 + \cdot\cdot\cdot + P_n)$ is an α-fixed projection $\leqq E$, and our assumptions force $E = P_1 + \cdot\cdot\cdot + P_n$. In the same fashion, it follows that if $P_{n+1} \neq 0$ for all n, then $E = \sum_{n=1}^{\infty} P_n$.

Define $Q_n = P_n - \alpha P_{n+1}$ $(n \geqq 1)$. We have $P_0 = \sum_{n \geqq 1} \alpha^n Q_n$. Moreover, the projections $\alpha E, Q_1, Q_2, \cdot\cdot\cdot, I - (E + P_0)$ are mutually orthogonal with LUB I, as are the projections E, $\alpha Q_1, \alpha^2 Q_2, \cdot\cdot\cdot, I - (E + P_0)$. Therefore, by Lemma 3.1, the formula

$$\beta(P) = \alpha(E)\alpha(P) + \sum_n Q_n \alpha^{-n}(P) + [I - (E + P_0)]P \tag{5.12}$$

defines an element of $[G]$ which agrees on E with α. Now if $R \leqq Q_n$, then $\beta^k R = \alpha^k R$ $(k \leqq n)$ and $\beta^{n+1} R = \beta(\alpha^n R) = Q_n R = R$. This shows that the projection $R_1 = \sum_{k=0}^{n} \beta^k R$ is β-fixed and $RQ_n = R_1 Q_n$. Therefore, Q_n is abelian relative to β. Therefore, all projections $\alpha^i Q_n$ $(i \leqq n)$ are abelian relative to β. But $\sum_{n \geqq 1} \sum_{k=0}^{n} \alpha^k Q_n = E + P_0$, and the complement of $E + P_0$ is automatically abelian relative to β. It follows that the group generated by β is of type I, and (5.11) is verified.

Turning to the theorem itself, we have noted in Section 2 that $[G]$ is the direct sum of a type I and a type II group, and in the present situation, it is clear that these summands are the full groups generated by restrictions of α. It will clearly suffice, therefore, to assume that G is of type II. This done,

denote by Z the fixed algebra of G, and corresponding to $\delta > 0$, choose a projection E in M such that $E_Z(E) = 1 - \delta$ (this by Maharam's lemma). By this construction, $\lambda(I - E) < \delta$, and E dominates no α-fixed projections (save 0). Apply (5.11) to this E. If K denotes the type I group generated by the β of (5.11), then

$$d(\alpha, [K]) \leqq d(\alpha, \beta) = \sup_P \lambda(\alpha P \Delta \beta P)$$
$$= \sup_P \lambda((\alpha P)(I - E) \Delta (\beta P)(I - E)) \leqq 2\lambda(I - E) < 2\delta.$$

If k is any positive integer, then the triangle inequality and the invariance of d give $d(\alpha^k, \beta^k) \leqq \sum_{r=0}^{k-1} d(\alpha^{k-r}\beta^r, \alpha^{k-(r+1)}\beta^{r+1}) < 2k\delta$, and $d(\alpha^{-k}, \beta^{-k}) = d(\alpha^k, \beta^k) < 2k\delta$. It follows therefore that for any $\epsilon > 0$ and any finite subset F of G, there exists a type I subgroup K of G such that $d(\gamma, [K]) < \epsilon$, for all $\gamma \in F$, and hence $\delta(\gamma, [K]) < 2\epsilon$. Thus condition (1) of Proposition 5.1 is verified, and by definition, G is approximately finite.

LEMMA 5.5. *For any MP automorphism α and arbitrary MP groups G_1 and G_2, $E([G_1], \alpha)E([G_2], \alpha) = E([G_1] \cap [G_2], \alpha)$.*

Proof. Write $E_i = E([G_i], \alpha)$, and choose α_i in $[G_i]$ such that $E_i\alpha_i = E_i\alpha$. We have $E_1E_2\alpha = E_1E_2\alpha_1 = E_1E_2\alpha_2$. The main step consists in showing that there exists a γ in $[G_1] \cap [G_2]$ such that $E_1E_2\gamma = E_1E_2\alpha_1 = E_1E_2\alpha_2$. To this end, write $E' = \alpha_1^{-1}(E_1E_2) = \alpha_2^{-1}(E_1E_2)$, and note that α_1 and α_2 agree on E'. Let E'_0 be the maximal α_1-fixed projection under E'. Let $E = E' - E'_0$, and construct the β of (5.11) for the pair (E, α_1). From the fact that $\alpha_1 = \alpha_2$ on E, it is easy to conclude that the formula (5.12) defining β in terms of α_1 remains valid when α_1 is replaced by α_2. It follows that $\beta \in [G_1] \cap [G_2]$. Now let $\gamma = \beta$ on $E \cup \alpha_1 E$, $\gamma = \alpha_1$ (and hence α_2) on E'_0, $\gamma =$ identity elsewhere. Then $\gamma \in [G_1] \cap [G_2]$ and $\gamma = \alpha_1 = \alpha_2$ on E'. Now for all P, $\gamma(E'P) = \alpha_1(E'P) = E_1E_2\alpha_2(P) = E_1E_2\alpha_2(P)$. This proves that $E_1E_2 \leqq E([G_1] \cap [G_2], \alpha)$. Automatically $E([G_1] \cap [G_2], \alpha)$ is dominated by each E_i. Therefore, $E([G_1] \cap [G_2], \alpha) = E_1E_2$, and the lemma is proved.

With this, it is easy to establish

THEOREM 2. *Any subgroup of an approximately finite group is itself approximately finite.*

Proof. Let G be an approximately finite group, and let K be any subgroup of $[G]$. Let F be a finite subset of K, and take $\epsilon > 0$. There exists a type I subgroup L of $[G]$ such that $\lambda(E([L], \alpha) > 1 - \epsilon$, for all α in

F. Applying Lemma 5.5 and the fact that $E([K],\alpha)=I$, we obtain $\lambda(E([K]\cap[L],\alpha))=\lambda(E([L],\alpha))>1-\epsilon$. Since any subgroup of a type I group is itself of type I, it follows that $[L]\cap[K]$ is a type I subgroup of $[K]$. This proves that K is approximately finite.

In particular, therefore, any local subgroup $[G]_P$ of the full group determined by an approximately finite group is approximately finite. Conversely,

COROLLARY 5.1. *Approximate finiteness is a local property: if G is an MP automorphism group with fixed algebra Z, and if for some projection P in M, $[G]_P$ is approximately finite, then so is $[G]_{\bar{P}}$, where $\bar{P}$ denotes the the Z-carrier of P.*

Proof. Plainly, we can assume that G is of type II and that $\bar{P}=I$. For each $\delta>0$, there exists a projection C in Z such that $\lambda(C)>1-\delta$ and $E_Z(P)$ is bounded away from 0 on C. Granting we have proved $[G]_C$ approximately finite, for each such C, then it will clearly follow that $[G]$ is itself approximately finite. Therefore, we can assume $E_Z(P)\geqq 1/n$, for some integer n. This done, apply Maharam's lemma to choose mutually orthogonal projections $P_0,\cdots,P_{n-1}$ such that $E_Z(P_i)=1/n$, for each i, whence $\sum_{i=0}^{n-1}P_i=I$. By Lemma 3.2, given i, there exists a ρ in $[G]$ such that $\rho P_i\leqq P$. It follows that $\rho[G]_{P_i}\rho^{-1}\subset[G]_P$, so by Theorem 2, $\rho[G]_{P_i}\rho^{-1}$ and therefore $[G]_{P_i}$ is approximately finite. Let

$$[G]_{P_0,\cdots,P_{n-1}}=[\beta\in[G]\mid\beta P_i=P_i \text{ for each } i].$$

This group will be approximately finite, being a direct sum of the approximately finite groups $[G]_{P_i}$ on P_iM. By Lemma 3.3, there exists an α in $[G]$ such that $\alpha P_i=P_{i+1}$ (indices mod n) and $\alpha^n=$ identity. Further, let $G_0=[\beta\in[G]_{P_0,\cdots,P_{n-1}}\mid\beta\alpha=\alpha\beta]$. For each γ in $[G]_{P_0,\cdots,P_{n-1}}$ and each i, the automorphism

$$\gamma'(Q)=\sum_{j=0}^{n-1}\alpha^j[P_i\gamma(\alpha^{-j}(Q))] \tag{5.13}$$

lies in G_0 and agrees with γ on all projections dominated by P_i. Therefore, $E([G_0],\gamma)\geqq P_i$ for all i, showing that $E([G_0],\gamma)=I$, $\gamma\in[G_0]$, and $[G_0]=[G]_{P_0,\cdots,P_{n-1}}$. Let K denote the group generated by α and G_0. Application of Lemma 3.5 shows that the fixed algebra Z_K of K coincides with Z. Given an arbitrary γ in $[G]$, Lemma 3.2 and the fact $Z=Z_K$ show that $[K]$ will contain an element μ such that $\mu^{-1}\gamma P_i=P_i$, for all i

simultaneously. Therefore, $\mu^{-1}\gamma \in [G]_{P_0,\cdots,P_{n-1}} = [G_0] \subset [K]$, so $\gamma \in [K]$, and $[G] = [K]$.

The problem is now reduced to showing that K is approximately finite. For this, it will suffice to prove the following: given $\beta_1, \cdots, \beta_r$ in G_0 and $\epsilon > 0$, there exists a finite subgroup S of $[G]$ containing α and containing element β''_i $(1 \leqq i \leqq r)$ such that $d(\beta_i, \beta''_i) < \epsilon$. To construct this S, choose a finite subgroup S_0 of $[G]_{P_0}$ containing elements β'_i $(1 \leqq i \leqq r)$ such that $\sup_{P \leqq P_0} \lambda(\beta_i(P) \Delta \beta'_i(P)) < \epsilon/n$. As in (5.13), set $\beta''_i = \sum_k \alpha^k [P_0 \beta'_i(\alpha^{-k}(\cdot))]$. Each β''_i commutes with α, and α and the β''_i generate a finite subgroup of $[G]$. Now, for any P in M,

$$\lambda(\beta''_i(P)\Delta\beta_i(P)) = \sum_k \lambda(\alpha^k(P_0\beta'_i(\alpha^{-k}P))\Delta\alpha^k(P_0\beta_i(\alpha^{-k}P)))$$
$$= \sum_k \lambda(\beta'_i(P_0\alpha^{-k}P)\Delta\beta_i(P_0\alpha^{-k}P)) < n(\epsilon/n) = \epsilon.$$

This concludes the proof.

6. The structure of approximately finite groups. As noted earlier, a restricted direct product $\prod_{a \in A} G_a$ of groups of order two is approximately finite in any faithful free action. One such action, of particular significance, can be realized as follows. Consider each G_a as a two-point measure space, and form the measure space (S, m) consisting of the direct product of these two-point measure spaces and an arbitrary finite measure space (the latter to appear as a fixed algebra). The generator of each G_a determines an involution of (S, m), and together these involutions generate a group of MP transformations of (S, m) which, algebraically, is isomorphic to the original product group. Passage to the hyperstonian measure space associated with (S, m) represents this group, in the customary setting of this theory, as a group of MP automorphisms. Now essentially, our main objective is to show that this example is canonical: subject to certain natural countability conditions, an arbitrary type *II* approximately finite automorphism group is equivalent to the automorphism group arising in the above construction with A countably infinite. [In this equivalence theory, to avoid rather unrewarding complications, we do not attempt to discuss cases involving either non-countable groups or measure algebras with bases of arbitrary cardinality. It is of interest that, in the parallel W^*-algebra theory of approximate finiteness, the difficulties atending the non-separable cases have been resolved by Misonou [7].]

For notation in the following, G will denote a type *II* group of MP automorphisms of (M, λ) with fixed algebra Z.

LEMMA 6.1. *Assume G approximately finite. Let K be a finite freely acting subgroup of $[G]$, and α an arbitrary element of $[G]$. Then, given $\epsilon > 0$, there exists a type I_n subgroup L of $[G]$ such that $K \subset [L]$ and $\lambda(E([L],\alpha)) > 1 - \epsilon$.*

Proof. Corresponding to $\delta > 0$ (to be specified later), choose a bounded type I subgroup L_1 of $[G]$ such that $\lambda(E([L_1],\gamma)) > 1 - \delta$, where γ runs over the finite set $F = \{K, \alpha\}$.

Let Z_K be the fixed algebra of K, and $n =$ order K. We assert: if P is any projection with $\lambda(I - P) < \delta$, then there exists a projection C in Z_K, $C \leqq P$, such that $\lambda(I - C) < n\delta$. To see this, let $P_1, \cdots, P_n$ be an abelian base for K over Z_K, and write $P = \sum_i C_i P_i$ $(C_i \in Z_K)$. Let $C = \prod_{i=1}^n C_i$. Then,

$$\lambda(C) \geqq 1 - \sum_{i=1}^n \lambda(I - C) = -(n-1) + n[(1/n) \sum_i \lambda(C_i)]$$
$$= 1 - n\lambda(I - P) > 1 - n\delta,$$

as asserted.

Define $E = \prod_{\gamma \in F} E([L_1],\gamma)$. Clearly, $\lambda(E) > 1 - (n-1)\delta$. By the above, we can choose a C in Z_K such that $C \leqq E$ and $\lambda(C) > 1 - n(n+1)\delta$. For each γ in F, there exists a γ_1 in $[L_1]$ such that $C\gamma(P) = C\gamma_1(P)$, for all P. In particular, $C\alpha(P) = C\alpha_1(P)$, α_1 in $[L_1]$. We claim, an α_2 in $[L_1]$ exists such that α_2 leaves $I - C$ absolutely fixed and $D\alpha_1 = D\alpha_2$, for some $D \leqq C$, with $\lambda(D) > 1 - 2n(n+1)\delta$. In fact, set $D = C\alpha_1(C)$. Now

$$E_{Z_{L_1}}(C\alpha_1(I - C)) = E_{Z_{L_1}}((\alpha_1 C)(I - C)),$$

so by Lemma 3.2, $[L_1]$ contains a ρ such that $\rho[(\alpha_1 C)(I - C)] = C\alpha_1(I - C)$, $\rho^2 =$ identity, and $\rho =$ identity off $C \Delta \alpha_1 C$. Then

$$(\rho\alpha_1)C = \rho[(\alpha_1 C)(I - C) + (\alpha_1 C)C] = C\alpha_1 C + C\alpha_1(I - C) = C,$$

and if we set $\alpha_2 = \rho\alpha_1$ on C, $\alpha_2 =$ identity on $(I - C)$, then $D\alpha = D\alpha_1 = D\alpha_2$ and $\lambda(D) > -1 - 2\lambda(I - C) > 1 - 2n(n+1)\delta$. Denote by L_2 the group generated by K and α_2. This group has a fixed algebra containing $(I - C)Z_K + CZ_{L_1}$, which is of bounded type I, so therefore L_2 itself is a bounded type I group. Now $E([L_2],\alpha) \geqq D$, and if we take $\delta < \epsilon/2n(n+1)$, it follows that $\lambda(E([L_2],\alpha)) > 1 - \epsilon$. By Lemma 4.5, for an appropriate integer n, there exists a type I_n subgroup L of $[G]$ such that $[L_2] \subset [L]$. Automatically, $\lambda(E[L],\alpha)) > 1 - \epsilon$, and the lemma is proved.

As we have noted in Lemma 4.3, any finite freely acting group K has an

abelian basis of the form $[kP \mid k \in K]$, and conversely, if P is any projection such that the kP are mutually orthogonal and $\sum kP = I$, then the kP form an abelian basis for K. By a *basis algebra* $\mathfrak{B}$ for K we mean the finite K-invariant boolean algebra generated by an abelian basis of the form kP $(k \in K)$. By a generator of such an algebra we mean any one of its atoms, that is, any kP. In the following we speak of *couples* $(K, \mathfrak{B})$, where K is understood to be a finite freely acting subgroup of $[G]$ and $\mathfrak{B}$ a basis algebra for K.

Definition 6.1. A set of couples $(K_i, \mathfrak{B}_i)$ $(1 \leqq i \leqq n)$ is said to be *mutually independent* if, for each pair i, j with $i \neq j$, K_i lies in the centralizer of K_j and $\mathfrak{B}_i$ is contained in the fixed algebra Z_j of K_j.

It is clear, of course, that the $(K_i, \mathfrak{B}_i)$ are mutually independent if and only if they are pairwise independent.

LEMMA 6.2. *If the couples* $(K_i, \mathfrak{B}_i)$ $(1 \leqq i \leqq n)$ *are mutually independent, then* 1) *the group* K *generated by the* K_i *is freely acting and algebraically is a direct product* $K_1 \times K_2 \times \cdots \times K_n$, 2) $E_{Z_1 \cap \cdots \cap Z_n} = E_{Z_1} E_{Z_2} \cdots E_{Z_n}$, *and* 3) *if* $Q_i \in \mathfrak{B}_i$, *then*

$$\lambda(Q_1 \cdots Q_n) = \lambda(Q_1)\lambda(Q_2) \cdots \lambda(Q_n).$$

Proof. (1) Say $k_1 k_2 \cdots k_n$ leaves a non-zero projection P absolutely fixed $(k_i \in K_i)$. For each i, P dominates a non-zero projection of the form CP_i (P_i a generator of $\mathfrak{B}_i$, C in Z_i), and because $\mathfrak{B}_i \subset Z_j$ $(j \neq i)$, we have $CP_i = (k_1 \cdots k_n)(CP_i) = (k_1 \cdots k_n)(C) k_i(P_i)$. This implies $k_i = e$, for otherwise $CP_i = CP_i k_i P_i = 0$, a contradiction. Therefore, $k_1 = \cdots = k_n = e$, and K is freely acting. A fortiori, $k_1 k_2 \cdots k_n = e$ entails $k_1 = \cdots = k_n = e$, and it follows that K is a direct product $K_1 \times \cdots \times K_n$.

(2) Each k in K_i implements an automorphism of Z_j $(j \neq i)$, since by assumption k lies in the centralizer of K_j. By (2.6), therefore, $kE_{Z_j} = E_{Z_j} k$, and we have

$$kE_{Z_1} \cdots E_{Z_n}(P) = E_{Z_1} \cdots E_{Z_{j-1}} k E_{Z_j} \cdots E_{Z_n}(P) = E_{Z_1} \cdots E_{Z_n}(P).$$

It follows that $E_{Z_1} \cdots E_{Z_n}(P)$ lies in the fixed algebra $Z = Z_1 \cap \cdots \cap Z_n$ of K. Plainly $\lambda[C(E_{Z_1} \cdots E_{Z_n}(P))] = \lambda(CP)$, for each C in Z and P in M, and by the uniqueness of E_Z (remark following (2.1)), we have $E_Z = E_{Z_1} \cdots E_{Z_n}$, as asserted. The statement (3) of the lemma follows readily from the fact that Q_i in $\mathfrak{B}_i$ entails $E_{Z_j}(Q_i) = Q_i$ $(j \neq i)$ and $E_{Z_i}(Q_i) = \lambda(Q_i)$.

LEMMA 6.3. *Assume G approximately finite. Let $(K, \mathcal{B}_K)$ be a given couple, and let α be an arbitrary element of $[G]$. Then, for each $\epsilon > 0$, there exists a couple $(L, \mathcal{B}_L)$ independent of $(K, \mathcal{B}_K)$, where L is a cyclic group with order a power of two, and $\lambda(E([K \times L], \alpha)) > 1 - \epsilon$.*

Proof. Choose $\delta > 0$ (to be specified later) and apply Lemma 6.1: there exists a type I_m subgroup L_1 of $[G]$ such that $K \subset [L_1]$ and $\lambda(E([L_1], \alpha)) > 1 - \delta$. If $n =$ order K, then n divides m and we can write $m = nt$. Choose an integer a such that $t/2^a < \delta$, and then choose an integer b such that $b/2^a \leqq 1/t < (b+1)/2^a$. One has $1 \geqq bt/2^a > 1 - t/2^a > 1 - \delta$. The fixed algebra Z of G is of type II in Z_{L_1}, and so by Maharam's lemma, there exists a projection C in Z_{L_1} such that $E_Z(C) = bt/2^a$. Let P be a generator of $\mathcal{B}_K$, so $E_{Z_K}(P) = 1/n$, and $E_Z(CP) = E_Z(E_{Z_K}(CP)) = E_Z(CE_{Z_K}(P)) = bt/n2^a$. Now $[L_1]_C$ is a type I_{nt} subgroup of $[G]_C$ on CM. By Lemma 4.5 (first part of proof), we can find a subgroup L_2 of $[G]_C$ on CM of type I_{nbt} such that $[L_1]_C \subset [L_2]$. Now, we claim, CP can be partitioned $CP = P_0 + \cdots + P_{bt-1}$, where the P_i are part of an abelian basis for L_2. In fact, the Z_{L_2}-carrier of CP is C, so CP dominates an abelian projection P_0 of L_2 with Z_{L_2}-carrier C. One has $E_{Z_{L_2}}(PC - P_0) = (1/n - 1/nbt)C$, so $PC - P_0$ is either 0 or again has Z_{L_2}-carrier C. In the latter case, $PC - P_0$ dominates an abelian projection P_1 of L_2 with Z_{L_2}-carrier C. Iteration of argument will lead to the ascribed partition. By the familiar method, choose a ρ in $[L_2]$ such that $\rho(P_i) = P_{i+1}$ (indices mod bt), $\rho^{bt} =$ identity, and $\rho =$ identity off PC.

Turning to $(I - C)$, we have $E_Z(P(I-C)) = (2^a - bt)/n2^a$, and by Maharam's lemma, we can partition $P(I - C) = Q_0 + \cdots + Q_{2^a - bt - 1}$ as a sum of mutually orthogonal projections each with expectation $1/n2^a$ relative to Z. Choose a β in $[G]$ which sends Q_i into Q_{i+1} (indices mod $2^a - bt$) and has order $2^a - bt$. Next, choose a γ in $[G]$ such that $\gamma(P_{bt-1}) = Q_0$, $\gamma^2 =$ identity. Define τ in $[G]$ as follows: first, set $\tau_1 = \rho$ on P_i $(i < bt - 1)$, $\tau_1 = \gamma$ on P_{bt-1}, $\tau_1 = \beta$ on Q_i $(i < 2^a - bt - 1)$, and $\tau_1 = \rho\gamma\beta$ on $Q_{2^a - bt - 1}$; and then define $\tau(Q) = k\tau_1 k^{-1}(Q)$ for $Q \leqq kP$ (k in K). It follows readily that τ generates a freely acting cyclic group L on M of order 2^a, that τ commutes with K, and that $\mathcal{B}_K \subset Z_L$. Take $\sum_{k \in K} kP_0$ as the generator of a basis algebra $\mathcal{B}_L$ of L. Then $\mathcal{B}_L \subset Z_K$, and $(K, \mathcal{B}_K)$, $(L, \mathcal{B}_L)$ are independent couples. Since the automorphism ρ agrees on $P_0 + \cdots + P_{bt-2}$ with τ, on P_{bt-1} with τ^{1-bt}, and is the identity elsewhere, we have $\rho \in [L]$. Therefore, the group K_0 on CM generated by ρ (on CM) and $[K]_C$ lies in $[K \times L]_C$.

Also $K_0 \subset [L_2]$, and that equality $[K_0] = [L_2]$ holds follows from the readily verified fact that R in Z_{K_0} entails R in Z_{L_2}. Therefore, $[L_2] = [K_0] \subset [K \times L]_C$. Already, $[L_1]_C \subset [L_2]$, and we recall that $\lambda(E([L_1], \alpha)) > 1 - \delta$, namely, $[L_1]$ contains an α_1 which agrees with α on a projection of measure $> 1 - \delta$. The automorphism α_2 defined to be α_1 on C and the identity elsewhere therefore agrees with α on a projection of measure $> 1 - 2\delta$, and $\alpha_2 \in [K \times L]$. We have $\lambda(E([K \times L], \alpha)) > 1 - 2\delta$. Take $\delta < \epsilon/2$, and the lemma follows.

LEMMA 6.4. *Let $(K, \mathcal{B})$ be a couple, with cyclic of order 2^n. Then there exist mutually independent couples $(K_i, \mathcal{B}_i)$ such that K_i has order 2, $[K] = [K_1 \times \cdots \times K_n]$, $\mathcal{B}$ is generated by the $\mathcal{B}_i$, and any couple $(L, \mathcal{C})$ independent of $(K, \mathcal{B})$ is independent of each $(K_i, \mathcal{B}_i)$.*

Proof. Set $\Lambda = (0, 1, \cdots, 2^n - 1)$. For $1 \leqq m \leqq n$, define S_m as $[i \text{ in } \Lambda \mid (j-1)2^{n-m} \leqq i < j2^{n-m}$, for some odd integer $j]$. Let P be a generator of $\mathcal{B}$ and α a generator of K. Define $R_m = \sum_{i \in S_m} \alpha^i P$, and define $\alpha_m = \alpha^{2^{m-n}}$ on R_m, $\alpha_m = \alpha^{-2^{m-n}}$ on $I - R_m$. Because $S_m \pm 2^{n-m} = \Lambda - S_m \pmod{2^n}$, we have $\alpha_m(R_m) = I - R_m$, $\alpha_m{}^2 =$ identity. Therefore α_m is an automorphism in $[K]$. Moreover, if $p \neq m$, then $S_m \pm 2^{n-p} = S_m \pmod{2^n}$, whence $\alpha_p(R_m) = R_m$. Now for any Q and $p \neq m$,

$$\alpha_m \alpha_p Q = \alpha^{2^{n-m} + 2^{n-p}}(Q)(I - R_m)(I - R_p) + \alpha^{2^{n-m} - 2^{n-p}}(Q) R_p (I - R_m)$$
$$+ \alpha^{-2^{n-m} + 2^{n-p}}(Q)(I - R_p) R_m + \alpha^{-2^{n-m} - 2^{n-p}}(Q) R_m R_p,$$

which is symmetric in p and m. Therefore, α_p and α_m commute. If we set $K_m = \{e, \alpha_m\}$ and $\mathcal{B}_m = \{R_m, I - R_m\}$, it follows that the couples $(K_i, \mathcal{B}_i)$ are mutually independent. Now if G_1 and G_2 are type I_s groups and if $G_1 \subset [G_2]$, then $[G_1] = [G_2]$. Applying this fact here, we have $[K] = [K_1 \times \cdots \times K_n]$. The boolean algebra $\mathcal{B}_0$ generated by the $\mathcal{B}_i$ is K_i-invariant and the projection $P = R_1 \cdots R_m$ is an atom in $\mathcal{B}_0$. Because $K_1 \times \cdots \times K_n$ is freely acting and P is abelian for $[K] = [K_1 \times \cdots \times K_n]$, it follows that P is orthogonalized by $K_1 \times \cdots \times K_n$ (Lemma 4.1). Therefore $I = \sum_{k \in K_1 \times \cdots \times K_n} kP$, all kP lie in $\mathcal{B}_0$, and accordingly, $\mathcal{B}_0$ is a basis algebra for $K_1 \times \cdots \times K_n$. But $\mathcal{B}_0$ is a subalgebra of $\mathcal{B}$ having the same number of atoms, whence $\mathcal{B}_0 = \mathcal{B}$. Let $(L, \mathcal{C})$ be a couple independent of $(K, \mathcal{B})$. For each i, we have $\mathcal{C} \subset Z_K \subset Z_{K_i}$, and $\mathcal{B}_i \subset \mathcal{B} \subset Z_L$. Finally, if $\gamma \in L$, then $\gamma R_m = R_m$, and for any Q,

$$(\gamma \alpha_m)(Q) = \gamma[\alpha^{2^{n-m}}(Q)(I - R_m) + \alpha^{-2^{n-m}}(Q) R_m]$$
$$= \alpha^{2^{n-m}}(\gamma Q)(I - R_m) + \alpha^{-2^{n-m}}(\gamma Q) R_m = (\alpha_m \gamma)(Q).$$

Therefore, L lies in the centralizer of K_i. It follows that $(L, \mathcal{C})$ is independent of each $(K_i, \mathcal{B}_i)$, proving the lemma.

In the following, we call an automorphism group G countably generated if it is equivalent to a countable automorphism group. [If G is freely acting and countably generated, then it is easy to see that G must be countable.]

THEOREM 3. *Let G be a countably generated type II group of MP automorphisms of (M, λ). In order that G be approximately finite, it is necessary and sufficient that G be equivalent to a freely acting automorphism group $K = \prod_{n=1}^{\infty} K_n$, K being the restricted direct product of a countably infinite set of freely acting groups K_n each of order two.*

Proof. As noted in Section 5, sufficiency is clear. To prove necessity, we can suppose that the approximately finite group G is countably infinite. Let α_n be a sequence such that each $\alpha_n \in G$ and each g in $G = \alpha_n$, for infinitely many n. By repeated application of Lemma 6.3, we can construct a sequence of couples $(L_n, \mathcal{B}_n)$ with the following properties: $L_n \subset [G]$, order $L_n = 2^{r_n}$, $(L_n, \mathcal{B}_n)$ is independent of $(L_i, \mathcal{B}_i)$, for $1 \leqq i < n$, and $\lambda(E([L_1 \times \cdots \times L_n], \alpha_n)) < 1/n$. For each $(L_n, \mathcal{B}_n)$, select couples $(L_i^{(n)}, \mathcal{B}_i^{(n)})$ $(1 \leqq i \leqq r_n)$ satisfying the conclusion of Lemma 6.4. Denote by $(K_1, \mathcal{C}_1), (K_2, \mathcal{C}_2), \cdots$ the sequence $(L_1^{(1)}, \mathcal{B}_1^{(1)}), \cdots, (L_{r_1}^{(1)}, \mathcal{B}_{r_1}^{(1)})$, $(L_1^{(2)}, \mathcal{B}_1^{(2)}), \cdots, (L_{r_2}^{(2)}, \mathcal{B}_{r_2}^{(2)})$, $(L_1^{(3)}, \mathcal{B}_1^{(3)})$, etc. The $(K_i, \mathcal{C}_i)$ are pairwise independent. Therefore the group K generated by the K_i is freely acting and is the restricted direct product $K = \prod_{i=1}^{\infty} K_i$. For each n, $\lambda(E([K], \alpha_n)) < 1/n$, and therefore, for each g in G, $\lambda(E([K], g)) = 0$, giving $[G] \subset [K]$. Equality $[G] = [K]$ follows by construction of K, and the theorem is proved.

A fortiori, if the countably generated type *II* group G is approximately finite, then G is equivalent to a freely acting group. Is the same true when G is not assumed approximately finite? This question is not settled.

LEMMA 6.5. *Let G be a type II group with fixed algebra Z. Let $(K, \mathcal{B}_K)$ be a given couple, with $K \subset [G]$ and order $K = 2^m$, and let P be a projection in M. Then, for each $\epsilon > 0$, there exists a couple $(L, \mathcal{B}_L)$ independent of $(K, \mathcal{B}_K)$, with $L \subset [G]$ and order L a power of two, such that, if $\mathcal{B}$ denotes the boolean algebra generated by $\mathcal{B}_L$, $\mathcal{B}_K$ and Z_P, then $\inf_{C \in \mathcal{B}} \lambda(P \Delta C) < \epsilon$.*

Proof. Let R be a generator for $\mathcal{B}_K$, and let $\mathcal{C}$ denote the finite K-invariant boolean algebra generated by $\mathcal{B}_K$ and the translates kP $(k \in K)$ of P. As a member of $\mathcal{C}$, R is a sum of atoms $R = R_1 + \cdots + R_s$ of $\mathcal{C}$,

and any atom of $\mathcal{C}$ is a translate by K of one of these R_i. Select a $\delta > 0$ (to be specified later). Choose mutually orthogonal projections $D_1, \cdots, D_t$ in Z with $\sum_i D_i = I$ and scalars $a_{ij} \geqq 0$ $(1 \leqq i \leqq s, 1 \leqq j \leqq t)$ such that

$$E_Z(R_i) - \delta/2 \leqq \sum_{j=1}^{t} a_{ij} D_j \leqq E_Z(R_i) \qquad (1 \leqq i \leqq s). \tag{6.1}$$

(That this choice is possible becomes clear if one considers the functional representation $C(\Gamma_Z)$ of Z, Γ_Z being the spectrum of Z.) Moreover, it is clear that we can find a sufficiently large integer p such that, if $\rho = 2^{-(m+p)}$ and n_{ij} = the largest non-negative integer satisfying $n_{ij}\rho \leqq a_{ij}$, then

$$E_Z(R_i) - \delta \leqq \sum_{j=1}^{t} n_{ij}\rho D_j \leqq E_Z(R_i) \qquad (1 \leqq i \leqq s). \tag{6.2}$$

Now fix the pair (i, j) with $n_{ij} \neq 0$, and applying Maharam's lemma, choose mutually orthogonal projections $R_{ik}^{(j)} \leqq R_i D_j$ such that $E_Z(R_{ik}^{(j)}) = \rho D_j$ $(1 \leqq k \leqq n_{ij})$. Set $R_{i0}^{(j)} = R_i D_j - \sum_k R_{ik}^{(j)}$ $(n_{ij} \neq 0)$ and $R_{i0}^{(j)} = R_j D_j$ $(n_{ij} = 0)$. From (6.2), we infer that

$$E_Z(R_{i0}^{(j)}) < \delta D_j \qquad (1 \leqq j \leqq t). \tag{6.3}$$

Now $E_Z(D_j R) = D_j E_Z(R) = D_j/2^m = 2^p \rho D_j$, and $D_j R = \sum_{i=1}^{s} D_j R_i = \sum_{i=1}^{s} \sum_{k=0}^{n_{ij}} R_{ik}^{(j)}$, where (by (6.3)) $E_Z(\sum_{i=1}^{s} R_{i0}^{(j)}) < s\delta D_j$. Therefore, $D_j R$ is partitioned into a sum of $N_j = \sum_{i=1}^{s} n_{ij}$ mutually orthogonal projections $R_{ik}^{(j)}$ $(k > 0)$, each with expectation ρD_j relative to Z, and the complement of this sum in $D_j R$ has Z-expectation $(2^p - N_j)\rho D_j < s\delta D_j$. Denote these $R_{ik}^{(j)}$ $(k > 0)$ by $S_1^{(j)}, \cdots, S_N^{(j)}$, and again by Maharam's lemma, choose mutually orthogonal projections $S_{N_j+1}^{(j)}, \cdots, S_{2^p}^{(j)}$, each with expectation ρD_j relative to Z, such that $D_j R = \sum_{k=1}^{2^p} S_k^{(j)}$. Let $S_k = \sum_{j=1}^{t} S_k^{(j)}$ $(1 \leqq k \leqq 2^p)$. Then $R = \sum S_k$ and $E_Z(S_k) = \rho$. Choose an $\alpha \in [G]$ such that $\alpha(S_k) = S_{k+1}$, $(1 \leqq k \leqq 2^p - 1)$, $\alpha(S_{2^p}) = S_1$, α^{2^p} = identity, α = identity off R. Next, define β in $[G]$ as follows: $\beta(Q) = k\alpha k^{-1}(Q)$ when $Q \leqq kR$ $(k \in K)$. This automorphism β commutes with each k in K and generates a freely acting cyclic group L of order 2^p. Let $Q = \sum_{k \in K} kS_1$. Then each $\beta^i Q \in Z_K$, and

$$\sum_{i=1}^{2^p} \beta^i Q = \sum_{k \in K} k[\sum_{i=1}^{2^p} \beta^i S_1] = \sum_{k \in K} kR = I.$$

Thus, the $\beta^i Q$ generate a basis algebra $\mathcal{B}_L$ for L, and $\mathcal{B}_L \subset Z_K$. By construction, $\mathcal{B}_K \subset Z_L$. It follows that $(L, \mathcal{B}_L)$ and $(K, \mathcal{B}_K)$ are independent couples, and L has order a power of two.

We return to the given P in $\mathcal{C}$. This P has the form $P = \sum_{k \in K} kE_K$, where each $E_k \leqq R$ and is 0 or a sum of R_i's. Denote by F_k the sum of all projections of the form $R_{ik}^{(j)}$ $(k > 0)$ which are dominated by E_k. Each of these $R_{ik}^{(j)}$ lies in the boolean algebra $\mathcal{B}$ generated by $\mathcal{B}_K$, $\mathcal{B}_L$ and Z_P: for $QR = S_1$, so all S_k lie in the algebra generated by $\mathcal{B}_K$ and $\mathcal{B}_L$, and the $R_{ik}^{(j)}$ are obtained as slices of the S_k by elements of Z_P. For any $R_i \leqq E_k$, we have $E_Z(R_i - R_iF_k) < \delta$ (by (6.3)), so $E_Z(E_k - F_k) < s\delta$, and $E_Z(P - \sum_{k \in K} kF_k) < 2^m s\delta$. But $\sum_{k \in K} kF_k \in \mathcal{B}$. If we take $\delta < \epsilon/s2^m$ (and note that m and s depend only on K and P), the lemma follows.

We say that (M, λ) is countably generated over a hyperstonian subalgebra Z if there exists a sequence P_n in M_P such that the boolean algebra generated by the P_n and Z_P is dense in M_P relative to the metric $\lambda(P \Delta Q)$. [This is equivalent to the following: there exists a sequence A_n in M such that the *-algebra generated by the A_n and Z is dense in M relative to the norm $\lambda[(A - B)(A - B)^*]^{\frac{1}{2}}$.]

THEOREM 4. *Let G be a type II group of MP automorphisms of (M, λ) with fixed algebra Z. Assume that (M, λ) is countably generated over Z. Then $[G]$ contains an approximately finite subgroup K with the same fixed algebra Z, and which is maximal among such subgroups of $[G]$.*

Proof. Let P_n be a sequence of projections in M which generate M over Z in the above sense. Let Q_n be a sequence of projections such that each Q_n lies in the sequence P_n and each P_n occurs infinitely often in the sequence Q_n. Repeated application of Lemma 6.5 yields a sequence of pairwise independent couples $(L_n, \mathcal{B}_n)$ with this property: for each n, the boolean algebra generated by $\mathcal{B}_1, \cdots, \mathcal{B}_n$ and Z_P contains a projection R_n such that $\lambda(R_n \Delta Q_n) < 1/n$. Let L be the freely acting approximately finite group $L = \prod_{n=1}^{\infty} L_n$, and let $\mathcal{B}$ be the boolean algebra generated by the $\mathcal{B}_n$. By construction, the boolean algebra $\mathcal{C}$ generated by $\mathcal{B}$ and Z_P is dense in M_P in the $\lambda(P \Delta Q)$ metric.

We claim, $Z = Z_L$, where Z_L is the fixed algebra of L. Take $P \in Z_L$. For each $\epsilon > 0$, there exist projections P_i in $\mathcal{B}$ and C_i in Z such that $\lambda(P \Delta \sum P_i C_i) < \epsilon$. Now any projection R in $\mathcal{B}$ (or for that matter, in the λ-closure $\bar{\mathcal{B}}$ of $\mathcal{B}$) is independent of Z_L, that is, has constant expectation relative to Z_L: this is obvious for projections in the boolean algebra generated by $\mathcal{B}_1, \cdots, \mathcal{B}_n$, since atoms in this algebra are abelian relative to $L_1 \times \cdots \times L_n$; and for any element R in $\bar{\mathcal{B}}$, we can find a sequence R_n such that $\lim_n \lambda(R \Delta R_n) = 0$ and R_n lies in the boolean algebra generated by

$\mathcal{B}_1, \cdots, \mathcal{B}_n$; $E_{Z_L}(R_n)$ is a constant c_n, so that $\lambda(R_n) = \lambda(E_{Z_L}(R_n)) = c_n$, and (using (2.7)) $E_{Z_L}(R) = \lim_n E_{Z_L}(R_n) = \lim \lambda(R_n) = \lambda(R)$, proving that R is independent of Z_L. Therefore, $E_{Z_L}(\sum P_i C_i) = \sum \lambda(P_i) C_i$ lies in Z, from which it follows that $P = E_{Z_L}(P) \in Z$. This shows that $Z_L \subset Z$. But since $L \subset [G]$, we must have $Z \subset Z_L$. Therefore $Z = Z_L$.

We have established the existence of approximately finite subgroups of $[G]$ having the same fixed algebra as G. To conclude, we wish to construct an approximately finite subgroup K of $[G]$ having fixed algebra Z and which is not contained in any larger approximately finite subgroup of $[G]$. The existence of K follows readily from Zorn's lemma. In fact, order full approximately finite subgroups of $[G]$ containing L by inclusion, and let L_a ($a \in A$) be a maximal linearly ordered subset. Let $K = [\bigcup_a L_a]$. K will of course have fixed algebra Z, and that K is approximately finite follows easily from the fact that, given α in K, there exists a finite set of indices $a_1, \cdots, a_n$ in A and elements g_i in L_{a_i} such that $\lambda(\bigcup_i F(\alpha, g_i)) > 1 - \epsilon$, for a preassigned $\epsilon > 0$; by linear order, these g_i will lie in some one L_a, and therefore $\lambda(E([L_a], \alpha)) > 1 - \epsilon$; application of the assumed approximate finiteness of the L_a then completes the proof.

The maximal approximately finite subgroup K of $[G]$ constructed in Theorem 4 will not in general be unique. In fact, it is not hard to show that, if $[G]$ contains precisely one maximal approximately finite subgroup, then G is already approximately finite and that subgroup coincides with $[G]$.

7. Weak equivalence of groups.

Definition 7.1. Let (M, λ) and (M', λ') be abstract non-atomic hyperstonian measure spaces. Let G (respectively, G') be a group of MP automorphisms of (M, λ) (respectively, (M', λ')). G and G' are called *weakly equivalent* if there exists an isomorphism (= *-isomorphism) φ of M on M' such that the transplant $\varphi^{-1} G' \varphi$ of G' is equivalent to G.

We do not require φ to be measure preserving, though it is clear that each $\varphi^{-1} g' \varphi$ ($g' \in G'$) will be a measure preserving automorphism of (M, λ), since any element of $[G]$ is automatically MP. If φ is not MP, then one can introduce a new measure λ_1 on M, equivalent to λ, and relative to which φ is MP and G remains an MP group; this fact is contained in the proof of the theorem to follow.

THEOREM 5. *Let G and G' be approximately finite type II group of MP automorphism of (M, λ) and (M', λ'), with fixed algebras Z and Z'. Assume that G and G' are countably generated, and that M (respectively M')*

is countably generated over Z (respectively Z'). Then in order that G and G' be weakly equivalent, it is necessary and sufficient that there exist an algebraic isomorphism θ of Z on Z'.

Proof. Necessity is trivial: if G and G' are weakly equivalent via an isomorphism φ, then the fixed algebra of $\varphi^{-1}G'\varphi$ is $\varphi^{-1}Z'$, so automatically $\varphi Z = Z'$. We turn to sufficiency. The main step here comes from iterated application of Lemmas 6.3-6.5, as in Theorem 3 and 4. One constructs a sequence of couples $(K_n, \mathfrak{B}_n)$ such that, first, each K_a is freely acting of order two and G is equivalent to the freely-acting product group $\prod_n K_n$, and second, if $\mathfrak{B}$ is the boolean algebra generated by the $\mathfrak{B}_n$ and $\mathfrak{C}$, the boolean algebra generated by $\mathfrak{B}$ and Z_P, then $\mathfrak{C}$ is dense in M_P in the λ-metric. (The procedure in the construction of the couples $(K_n, \mathfrak{B}_n)$ is to approximate alternately to elements of a generating set for G and a generating set for M over Z.) This construction applied to (M', λ') yields a sequence $(K'_n, \mathfrak{B}'_n)$ with the analogous properties.

Let $K = \prod_n K_n$, $K' = \prod_n K'_n$. It is evident that there exists an isomorphism $g \to g'$ of K on K' which sends K_n (qua subgroup of K) on K'_n, for each n. By the same token, it is easy to see that there exists a boolean isomorphism φ of $\mathfrak{B}$ on $\mathfrak{B}'$ such that

$$\varphi(kP) = k'\varphi(P), \qquad \theta(E_Z(P)) = E_{Z'}(\varphi(P)), \tag{7.1}$$

for all $k \in K$, $P \in \mathfrak{B}$; for recall that $\mathfrak{B}$ is the union of an increasing sequence of boolean algebras (these generated by the $\mathfrak{B}_i$, $1 \leqq i \leqq n$), each of whose atoms is an abelian projection for $K_1 \times \cdots \times K_n$. Now $\mathfrak{C}$ consists of projections of the form $P = \sum_{i=1}^{r} P_i C_i$ $(P_i \in \mathfrak{B}, C_i \in Z_P)$, in the representation of which we can assume the C_i are mutually orthogonal. If $P = \sum_{j=1}^{s} Q_j D_j$ is another such representation of P, then $P_j C_i D_j = Q_j C_i D_j$ for all i and j. From this it follows that $P_i = Q_j$ if $C_i D_j \neq 0$: in fact, if $\mathfrak{B}$ contains a non-zero projection R such that $RP_i = R$ and $RQ_j = 0$, then $RC_iD_j = 0$, so $E_Z(R)C_iD_j = 0$, so $C_iD_j = 0$, because $E_Z(R)$ is a constant. Therefore, $\varphi(P_i) = \varphi(Q_j)$ if $\theta(C_i)\theta(D_j) \neq 0$, and $\varphi(P_i)\theta(C_iD_j) = \varphi(Q_j)\theta(C_iD_j)$ holds in all cases. This show that $\sum \varphi(P_i)\theta(C_i) = \sum \varphi(Q_j)\theta(D_j)$. If we now define $\varphi(P) = \sum \varphi(P_i)\theta(C_i)$ (for C_i mutually orthogonal), then this definition is single-valued, and φ is plainly a boolean isomorphism of $\mathfrak{C}$ on $\mathfrak{C}'$. It is clear, moreover, that the formula (7.1) now holds for all P in $\mathfrak{C}$. Note that this formula shows that $P \in Z$ if and only if $\varphi(P) \in Z'$, and that $\theta(P) = \varphi(P)$, for P in Z.

We now introduce an equivalent measure λ_1 on M relative to which φ is MP. First of all, for A in Z, set $\lambda_0(A)=\lambda'(\theta(A))$. Then λ_0 is a faithful measure on Z which is normal, this because θ and λ' are normal. Next, for A in M, define $\lambda_1(A)=\lambda_0(E_Z(A))$. Clearly λ_1 is a measure. It is faithful, for if $A\geqq 0$ and $\lambda_1(A)=0$, then $\lambda_0(E_Z(A))=0$, so $E_Z(A)=0$, $A=0$. It is normal: if the A_a are bounded and SA, then

$$\begin{aligned}\mathrm{LUB}_a\lambda_1(A_a)&=\mathrm{LUB}_a\lambda_0(E_Z(A_a))=\lambda_0(\mathrm{LUB}_aE_Z(A_a))\\&=\lambda_0(E_Z(\mathrm{LUB}_aA_a))=\lambda_1(\mathrm{LUB}_aA_a).\end{aligned}$$

For any α in $[K]$, $\lambda_1(\alpha A)=\lambda_0(E_Z(\alpha A))=\lambda_0(E_Z(A))=\lambda_1(A)$, so $[K]$ remains an MP group. Now $\lambda_1(P)=0$ if and only if $P=0$ if and only if $\lambda(P)=0$, for any projection P, so λ_1 is equivalent to λ. Finally, φ is MP relative to λ_1: $\lambda_1(P)=\lambda_0(E_Z(P))=\lambda'(\theta(E_Z(P))=\lambda'(E_Z(P))=\lambda'(\varphi(P))$. It follows that (M,λ_1) is hyperstonian, and that φ is an isometry of $\mathcal{B}$ on $\mathcal{B}'$ relative to the λ_1 and λ' metrics. Therefore, φ can be extended uniquely to an isomorphism of M on M', the conditions (7.1) now being valid for all P in M.

We have $\varphi^{-1}k'\varphi=k$ for each $k\in K$, so $\varphi^{-1}K'\varphi$ is trivially equivalent to K. By construction, K is equivalent to G, K' to G', and by the transitivity of equivalence, we see that $\varphi^{-1}G'\varphi$ is equivalent to G. This proves the theorem.

If G and G' are type I_n groups on M and M' with fixed algebras Z and Z', and if there exists an isomorphism θ of Z on Z', then θ extends to an isomorphism φ of M on M' such that $\varphi^{-1}G'\varphi=G$: in fact, we can assume (up to equivalence) that G and G' are freely acting cyclic groups of order n, with generator α and β, and with abelian bases $\alpha^i(P)$ and $\beta^i(Q)$; this done, define $\varphi(\sum_i D_i\alpha^iP)=\sum_i\theta(D_i)\beta^iQ$ $(D_i\in Z)$ to obtain the desired isomorphism φ. Now given any group G with fixed algebra Z, there exist uniquely determined projection $C_\infty, C_1, C_2,\cdots$ in Z such that G_{C_i} is of type I_i (i finite) or II $(i=\infty)$. Under a weak equivalence of G with a group G', one will have $\varphi(C_n)=C'_n$, where the C'_n determine the type summands of G'. It follows easily from this discussion that Theorem 5 remains valid for groups G and G' not necessarily of type II if one adjoins the condition that $\theta(C_n)=C'_n$, for all n.

8. The existence of non-approximately finite groups. The detection of non-approximately finite groups in this presentation, as in the Murray-von Neumann construction, hinges on the derivation of necessary conditions which cannot be realized for certain groups, this by virtue of their algebraic (opposed to action) characteristics. The necessary conditions developed here

(Lemma 8.1), apparently unlike the "property Γ" of Murray-von Neumann, also turn out to be sufficient conditions, and the further development of the theory profits from this circumstance.

LEMMA 8.1. *Let K be a countable freely acting group of MP automorphisms of M. Assume that K is approximately finite. Then for any finite subset $x_1, \cdots, x_n$ of K and each $\epsilon > 0$, there exists a function $x \to E_x$ from K to M_P with the following properties:* (1) $E_x E_{xy} = E_x x(E_y)$ *and* $E_e = I$; (2) $\lambda(\prod_{i=1}^{n} E_{x_i}) > 1 - \epsilon$; (3) $\sum_x E_x$ *is bounded.*

Proof. By Theorem 3, K is equivalent to a freely acting group G which is algebraically the union of an increasing sequence of finite subgroups. Given x_i in K $(1 \leqq i \leqq n)$ and $\epsilon > 0$, it is clear that we can find a finite subgroup S of G such that $\lambda(\prod_{i=1}^{n} E([S], x_i)) > 1 - \epsilon$. Let $E_x = E([S], x)$. We will show that this function has the ascribed properties.

By Lemma 3.1, each x in K has a representation

$$x(P) = \sum_{g \in G} Q(g, x) g(P), \tag{8.1}$$

where for each x, the sets $Q(g, x)$, respectively, $g^{-1}Q(g, x)$, are mutually orthogonal and have LUB I. Because G is freely acting, the coefficient projections $Q(g, x)$ in (8.1) are uniquely determined; and because K is freely acting, $Q(g, x)Q(g, y) = 0$ if $x \neq y$. Computing $x(y(P))$ from (8.1) and comparing terms with $(xy)(P)$, we get

$$Q(g, xy) = \sum_{h \in G} Q(h, x) h[Q(h^{-1}g, y)]. \tag{8.2}$$

Now (8.1) also shows that $F(x, g) = g^{-1}Q(x, g)$. But $E_x = \mathrm{LUB}_{g \in S}\, gF(x, g)$, so that

$$E_x = \sum_{g \in S} Q(g, x). \tag{8.3}$$

From this, we obtain $\sum_{x \in K} E_x = \sum_{g \in S} [\sum_{x \in K} Q(g, x)] \leqq$ order S. This establishes condition (3) of the lemma. To verify condition (1), we use (8.2) to compute $E_{xy} = \sum_{g \in S} \sum_{h \in G} Q(h, x) h[Q(h^{-1}g, y)]$, so that

$$E_x E_{xy} = \sum_{g, h \in S} Q(h, x) h[Q(h^{-1}g, y)];$$

further,

$$xE_y = \sum_{h \in G} Q(h, x) h[\sum_{g \in S} Q(g, y)] = \sum_{h \in G} \sum_{g \in S} Q(h, x) h[Q(g, y)],$$

so

$$E_x x E_y = \sum_{h, g \in S} Q(h, x) h[Q(g, y)] = \sum_{h, g \in S} Q(h, x) h[Q(h^{-1}g, y)] = E_x E_{xy}.$$

Obviously $E_e = I$, and the lemma is proved.

PROPOSITION 8.1. *Let K be a countable discrete group. Suppose that K contains a subset F and elements x, y, z such that* 1) $F \cup xF = K$, 2) $yF \cap zF = \phi$, *and* 3) $yF \cup zF \subset xF$. *Then, in any faithful representation as a freely acting group of MP automorphisms of a non-atomic hyperstonian measure space (M, λ), K is non-approximately finite.*

Proof. We assume that K has a freely-acting faithful approximately finite representation on (M, λ) and will arrive at a contradiction. Take $0 < \epsilon < 1/5$. Let E_x be the function on K to M_P with the properties (1)-(3) of Lemma 8.1 for the subset x, y, z and this ϵ: one has $\lambda(E_xE_yE_z) > 1 - \epsilon$, the function $T = \sum_{u \in K} E_u$ is bounded, and $E_uE_{uv} = E_u uE_v$, $E_e = I$. Let $A = \sum_{u \in F} E_u$. By conditions (2) and (3) in the proposition,

$$\sum_{u \in F}(E_{yu} + E_{zu}) = \sum_{u \in yF} E_u + \sum_{u \in zF} E_u \leqq \sum_{u \in xF} E_u. \tag{8.4}$$

Multiply both sides of (8.4) by $E_xE_yE_z$ and use the relations $E_yE_{yu} = E_yyE_u$, etc. Then

$$E_xE_yE_z[yA + zA] \leqq E_xE_yE_zx(A). \tag{8.5}$$

Now $E_vT = \sum_u E_vE_u = \sum_u E_vE_{vu} = \sum_u E_vvE_u = E_vv(T)$. T^{-1} exists, since $T \geqq I$, and we obtain

$$E_vT^{-1} = E_vv(T^{-1}). \tag{8.6}$$

Let $B = AT^{-1}$, and multiply both sides of (8.5) by T^{-1} to obtain

$$E_xE_yE_z[yB + zB] \leqq E_xE_yE_zx(B), \tag{8.7}$$

where now $0 \leqq B \leqq I$. Write $Q = E_xE_yE_z$. Since $\lambda(Q) > 1 - \epsilon$, we have

$$\begin{aligned} \lambda(B) \geqq \lambda(QxB) = \lambda(QyB) + \lambda(QzB) &\geqq \lambda(yB) + \lambda(zB) - 2\epsilon \\ &= 2\lambda(B) - 2\epsilon. \end{aligned}$$

This gives the inequality

$$\lambda(B) \leqq 2\epsilon. \tag{8.8}$$

On the other hand, using condition (1) of the proposition, $T \leqq \sum_{u \in F} E_u + \sum_{u \in F} E_{xu}$. Therefore, $E_xT \leqq E_xA + E_xxA$, and applying (8.6), we have

$$E_x \leqq E_xB + E_xxB. \tag{8.9}$$

Therefore, $\lambda(E_x) \leqq 2\lambda(B)$. But $\lambda(E_x) \geqq \lambda(Q) > 1 - \epsilon$, so

$$\lambda(B) \geqq (1 - \epsilon)/2. \tag{8.10}$$

Inequalities (8.8) and (8.10) are inconsistent when $\epsilon < 1/5$. The proposition is proved.

The conditions of this proposition are not hard to realize. For example,

let K be the free product of three groups each or order two. If x, y and z denote the generators, and F is the set of all words (in reduced form) which begin with x, then it is clear that the conditions (1)-(3) hold. Further, Lemma 2.1 shows that this group has a faithful representation as a freely acting MP automorphism group. Therefore,

COROLLARY 8.1. *There exist countable discrete groups which have no faithful representations as freely acting approximately finite MP automorphism groups. In particular, there exist non-approximately finite automorphism groups.*

By Theorem 3, the free product of $n \geqq 3$ groups of order two (or for that matter, the group of all measure preserving transformations of (M, λ)) will be non-approximately finite. The free group on $n \geqq 2$ generators also satisfies the conditions of Proposition 8.1. It is of interest that the free product of two groups of order two is approximately finite, together with all abelian groups; these facts will be established in the course of further development of the theory.

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SUR LES REPRÉSENTATIONS UNITAIRES DES GROUPES DE LIE NILPOTENTS. I.*

Par J. DIXMIER.

Introduction. Soient G un groupe localement compact, H un sous-groupe abélien fermé distingué de G, $\hat{H}$ le dual de H. Tout élément s de G définit un automorphisme $x \rightarrow sxs^{-1}$ de H, donc un automorphisme de $\hat{H}$. L'ensemble des transformés d'un point de $\hat{H}$ par G est une partie de $\hat{H}$ appelée orbite. Disons que H est *régulièrement contenu* dans G s'il existe une suite de parties boréliennes $E_1, E_2, \cdots$ de $\hat{H}$, stables pour G, telles que toute orbite soit l'intersection des E_i qui la contiennent. Cette notion est due à Mackey [6]. Nous démontrerons le théorème suivant:

THÉORÈME 1. *Soient G un groupe de Lie réel connexe nilpotent, H un sous-groupe abélien fermé distingué connexe de G. Alors, H est régulièrement contenu dans G.*

On sait que la conclusion du théorème est inexacte lorsqu'on remplace l'hypothèse "G nilpotent" par l'hypothèse "G résoluble." D'autre part, on verra que la conclusion du théorème est également en défaut si H n'est pas supposé connexe. Enfin, un exemple inédit de Mackey montre qu'on ne peut supprimer l'hypothèse "G connexe."

Mackey a montré [6] que la notion de sous-groupe régulièrement contenu joue un rôle important dans la recherche des représentations unitaires d'un groupe. Grâce à ses résultats, nous déduirons du Théorème 1 les théorèmes suivants:

THÉORÈME 2. *Soient G un groupe de Lie réel connexe nilpotent, H un sous-groupe abélien fermé distingué connexe de G, $s \rightarrow U_s$ une représentation unitaire factorielle continue de G dans un espace hilbertien. Il existe un sous-groupe fermé connexe $G' \supset H$ de G, et une représentation unitaire continue U' de G' par des opérateurs scalaires, tels que U soit unitairement équivalente à la représentation de G induite par U'.*

(Si U est irréductible, U' s'effectue nécessairement dans un espace de dimension 1.)

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* Received August 24, 1958.

THÉORÈME 3.[1] *Toute représentation unitaire continue d'un groupe de Lie réel connexe nilpotent est de type* I.

Avant d'aborder les démonstrations, précisons quelques notations:

1° Une application f d'un ensemble A dans un ensemble B est dite injective si elle transforme deux éléments distincts en éléments distincts, surjective si $f(A) = B$, bijective si elle est injective et surjective;

2° Soit R une relation d'équivalence sur un ensemble X, et soit X' une partie de X. On désigne par $R_{X'}$ la relation d'équivalence induite par R dans X';

3° Une relation d'équivalence R dans un espace topologique X est dite séparée si l'espace quotient X/R est séparé;

4° Quand nous parlons de variété algébrique, nous ne supposons pas l'irréductibilité. Soient V une variété algébrique, E un sous-ensemble de V. On note $\dim E$ la plus grande dimension des composantes irréductibles de l'adhérence de E dans V pour la topologie de Zariski (*cf.* [3] et [4] pour la topologie de Zariski).

1. Démonstration du Théorème 1.

LEMME 1.[2] *Soient V un variété algébrique complexe, G un groupe algébrique complexe irréductible opérant à gauche dans V, l'application $(x,s) \to sx$ de $V \times G$ dans V étant partout régulière. Considérons deux points de V comme équivalents s'ils sont transformés l'un de l'autre par un élément de G. Soit R la relation d'équivalence ainsi définie. Alors, il existe une suite finie $(V_i)_{0 \leq i \leq n}$ de variétés algébriques contenues dans V possédant les propriétés suivantes:*

a) $\phi = V_0 \subset V_1 \subset \cdots \subset V_n = V$;

b) *chaque V_i est stable pour G;*

c) *la relation d'équivalence $R_{V_{i+1}-V_i}$ est séparée pour la topologie ordinaire* $(i = 0, \cdots, n-1)$.

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[1] Ce théorème résulte aussi de mon article " Sur les représentations unitaires des groupes de Lie algébriques," Ann. de l'Institut Fourier, 7 (1957), pp. 315-328 (d'ailleurs rédigé postérieurement au présent mémoire). Mais la démonstration donnée ici est beaucoup plus élémentaire. Le théorème résulte également de [9].

[2] Depuis que ce mémoire a été rédigé, des résultats beaucoup plus généraux que le Lemme 1 ont été obtenus par C. Chevalley (cf. son Traité de Géométrie Algébrique, à paraître).

Démonstration. Toutes les notions topologiques utilisées dans la démonstration de ce lemme se réfèrent à la topologie de Zariski, sauf mention expresse du contraire.

Nous procéderons par récurrence sur la dimension complexe m de V, en supposant le lemme établi pour les variétés de dimension $< m$.

Soient $C_1, \cdots, C_p$ les composantes irréductibles de V. Nous allons montrer (ce qui est d'ailleurs bien connu) que G laisse stable chaque C_i. Soit A l'intersection de C_i avec les C_j d'indice $j \neq i$; c'est une partie fermée de C_i rare dans C_i. Il suffit donc de montrer que, si x est un point de C_i non dans A, on a $Gx \subset C_i$. Or l'adhérence B de Gx dans V est irréductible. L'égalité $B = (B \cap C_1) \cup (B \cap C_2) \cup \cdots \cup (B \cap C_p)$ prouve que l'un des $B \cap C_j$ est égal à B, d'où $B \subset C_j$; alors, $x \in C_j$; comme $x \notin A$, on a $j = i$, d'où $B \subset C_i$, ce qui prouve notre assertion.

Il en résulte facilement qu'il suffit de prouver le lemme lorsque V est irréductible, ce que nous supposerons désormais.

L'application $\phi: (x, s) \to (x, sx)$ de $V \times G$ dans $V \times V$ est partout régulière, et $D_1 = \phi(V \times G)$ est le graphe de R. L'adhérence D de D_1 dans $V \times V$ est une variété algébrique irréductible; soit n sa dimension. L'ensemble D_1 contient une partie relativement ouverte de D ([3], exposé 7, th. 3). Donc $D - D_1$ est contenu dans une partie fermée F de D de dimension $n' < n$.

Pour tout $x \in V$, on désignera par V_x l'ensemble des éléments de $V \times V$ dont la première coordonnée est x. Pour tout $x \in V$, on a $\dim(V_x \cap D) \geqq n - m$ ([3], exposé 8, th. 3). D'autre part (loc. cit.), il existe un ensemble rare E' de V tel que, pour $x \notin E'$, on ait $\dim(V_x \cap F) \leqq n' - m$. Si $x \notin E'$, on a donc $\dim(V_x \cap F) < n - m$. Soit E l'ensemble des points $x \in V$ tels que $\dim V_x \cap (D - D_1) \geqq n - m$. Ce qui précède montre que E est rare dans V.

Montrons que E est stable par G. Soit $s \in G$. Soit S l'application bijective et birégulière de $V \times V$ dans $V \times V$ qui transforme (x, y) en (sx, y). Il est clair que $S(D_1) = D_1$, donc que $S(D) = D$. D'autre part $S(V_x) = V_{sx}$, et par suite $S(V_x \cap (D - D_1)) = V_{sx} \cap (D - D_1)$. Donc, si $x \in E$, on a $sx \in E$, ce qui prouve notre assertion.

Montrons que, si $(u, v) \in D$ et $(v, w) \in D_1$, on a $(u, w) \in D$. Soit $s \in G$ tel que $w = sv$. Soit S' l'application bijective et birégulière de $V \times V$ dans $V \times V$ qui transforme (x, y) en (x, sy). Il est clair que $S'(D_1) = D_1$, donc que $S'(D) = D$. Donc $(u, w) = S'((u, v)) \in D$.

Soient $x \in V - E$, $y \in V - E$ des éléments tels que $(x, y) \in D$. Nous allons montrer que $(x, y) \in D_1$. Raisonnant par l'absurde supposons $(x, y) \notin D_1$.

Si $w \in V$ est tel que $(y, w) \in D_1$, on a $(x, w) \in D$ d'après ce qui précède, et $(x, w) \notin D_1$ (car les relations $(x, w) \in D_1$ et $(y, w) \in D_1$ entraîneraient $(x, y) \in D_1$, contrairement à l'hypothèse), donc $(x, w) \in D - D_1$; donc la deuxième projection $\mathrm{pr}_2(V_x \cap (D - D_1))$ contient $\mathrm{pr}_2(V_y \cap D_1)$; donc $\dim V_x \cap (D - D_1) \geqq \dim V_y \cap D_1$; comme $x \notin E$, on en déduit que $\dim V_y \cap D_1 < n - m$; par ailleurs, comme $y \notin E$, on a $\dim V_y \cap (D - D_1) < n - m$; donc $\dim V_y \cap D < n - m$, ce qui est absurde. On a donc bien prouvé que $(x, y) \in D_1$.

Il en résulte que le graphe de la relation d'équivalence R_{V-E} est $[(V-E) \times (V-E)] \cap D$, donc est fermé dans $(V-E) \times (V-E)$ (au sens de Zariski, donc au sens ordinaire). Par ailleurs, R_{V-E} est une relation d'équivalence ouverte au sens de la topologie ordinaire puisqu'elle est définie par un groupe d'homéomorphismes de $V - E$. Donc ([2], chap. I, § 9, th. 2), R_{V-E} est séparée pour la topologie ordinaire. Soit A l'adhérence de E dans V (au sens de Zariski). Alors, A est stable pour G, R_{V-A} est séparée pour la topologie ordinaire, et $\dim A < \dim V$. Il suffit maintenant d'appliquer à A, dont la dimension est $< m$, l'hypothèse de récurrence.

Les Lemmes 2 et 3 nous permettront d'utiliser le Lemme 1 dans le domaine réel et non plus dans le domaine complexe.

LEMME 2. *Soient W un espace vectoriel réel de dimension finie, u un endomorphisme nilpotent de W, W' la complexification de W. Considérons iu comme un endomorphisme de W'. Si $x \in W$, $y \in W$ sont tels que* $(\exp(iu)) \cdot x = y$, *on a* $x = y$.

Démonstration. Le lemme est évident si la dimension n de W est égale à 1. Procédant par récurrence, supposons le lemme établi pour les dimensions $< n$. Il existe une base $(e_1, \cdots, e_n)$ de W et des nombres réels $\lambda_1, \cdots, \lambda_{n-1}$ tels que $ue_j = \lambda_j e_{j+1}$ pour $j = 1, \cdots, n-1$, et $ue_n = 0$. Soient $x = \xi_1 e_1 + \cdots + \xi_n e_n$, $y = \eta_1 e_1 + \cdots + \eta_n e_n$, les ξ_j et les η_j étant réels. Soit $W_j = \boldsymbol{R} e_j + \boldsymbol{R} e_{j+1} + \cdots + \boldsymbol{R} e_n$ ($\boldsymbol{R}$ étant le corps des nombres réels). On a $u^2(W_1) \subset W_3$, donc $y = (\exp(iu)) \cdot x \equiv x + iux \equiv \xi_1 e_1 + (\xi_2 + i\lambda_1 \xi_1) e_2$ (mod. W_3). Donc $\lambda_1 \xi_1 = 0$. Si $\xi_1 = 0$, on a $x \in W_2$, $y \in W_2$, et W_2 est stable pour u, donc $x = y$ d'après l'hypothèse de récurrence. Si $\lambda_1 = 0$, $\boldsymbol{R} e_1$ et W_2 sont stables pour u, et la décomposition en somme directe $W = \boldsymbol{R} e_1 + W_2$ donne encore $x = y$ d'après l'hypothèse de récurrence.

LEMME 3. *Soient $\mathfrak{g}$ une algèbre de Lie réelle nilpotente de dimension finie, $\mathfrak{g}' = \mathfrak{g} + i\mathfrak{g}$ sa complexification, G' un groupe de Lie complexe connexe d'algèbre de Lie $\mathfrak{g}'$. Alors, tout élément de G' se met sous la forme* $(\exp iY)(\exp X)$, *où* $X \in \mathfrak{g}$, $Y \in \mathfrak{g}$.

Démonstration. Le lemme est évident si la dimension n de $\mathfrak{g}$ est égale à 1. Procédant par récurrence, supposons le lemme établi pour les dimensions $< n$. Il existe un idéal $\mathfrak{h}$ de dimension 1 de $\mathfrak{g}$; cet idéal est contenu dans le centre de $\mathfrak{g}$. Soient $\mathfrak{h}'$ la complexification de $\mathfrak{h}$, et H' le sous-groupe connexe de G' correspondant à $\mathfrak{h}'$. Alors, la complexification de $\mathfrak{g}/\mathfrak{h}$ s'identifie à $\mathfrak{g}'/\mathfrak{h}'$, et G'/H' est un groupe de Lie complexe connexe d'algèbre de Lie $\mathfrak{g}'/\mathfrak{h}'$. Soit $s \in G'$. D'après l'hypothèse de récurrence, il existe $X_1 \in \mathfrak{g}$, $Y_1 \in \mathfrak{g}$ tels que $(\exp iY_1)(\exp X_1)$ et s soient congrus modulo H'. Il existe donc $X_2 \in \mathfrak{h}$, $Y_2 \in \mathfrak{h}$ tels que $s = (\exp iY_1)(\exp X_1)(\exp iY_2)(\exp X_2)$. Comme X_2 et Y_2 sont dans le centre de $\mathfrak{g}$, ceci s'écrit $s = (\exp i(Y_1 + Y_2))(\exp(X_1 + X_2))$. D'où le lemme.

LEMME 4. *Soient $\mathfrak{g}$ une algèbre de Lie nilpotente réelle, V un espace vectoriel réel de dimension finie, ρ une représentation linéaire de $\mathfrak{g}$ dans V par des endomorphismes nilpotents, G un groupe de Lie réel connexe d'algèbre de Lie $\mathfrak{g}$, σ la représentation* (*qu'on suppose exister*) *de G dans V correspondant à ρ. Considérons deux points de V comme équivalents s'ils sont transformés l'un de l'autre par un automorphisme $\sigma(s)$, où $s \in G$. Soit R la relation d'équivalence ainsi définie. Alors, il existe une suite finie $(V_i)_{0 \leq i \leq n}$ de variétés algébriques réelles contenues dans V possédant les propriétés suivantes:*

a) $\phi = V_0 \subset V_1 \subset \cdots \subset V_n = V$;

b) *chaque V_i est stable pour G;*

c) *la relation d'équivalence $R_{V_{i+1}-V_i}$ est séparée* $(i = 0, \cdots, n-1)$.

Démonstration. Soient V', $\mathfrak{g}'$, ρ' les complexifications de V, $\mathfrak{g}$, ρ; soient G' le groupe de Lie complexe connexe simplement connexe d'algèbre de Lie $\mathfrak{g}'$, σ' la représentation de G' correspondant à ρ'. Considérons deux points de V' comme équivalents s'ils sont transformés l'un de l'autre par un automorphisme $\sigma'(s')$, où $s' \in G'$. Soit R' la relation d'équivalence ainsi définie. Il existe (Lemme 1) des variétés algébriques complexes $V'_0, \cdots, V'_n$ contenues dans V', avec $\phi = V'_0 \subset \cdots \subset V'_n$, stables pour G', telles que les relations d'équivalence $R'_{V'_{i+1}-V'_i}$ soient séparées. Alors, les $V_i = V'_i \cap V$ sont des variétés algébriques réelles stables pour G, telles que les relations d'équivalence $R'_{V_{i+1}-V_i}$ soient séparées. Il suffit donc de montrer que $R'_V = R$. Il est clair que deux points de V congrus modulo R sont aussi congrus modulo R'_V. Réciproquement, soient x, y deux points de V congrus modulo R'_V. Il existe $s' \in G'$ tel que $y = \sigma'(s')x$. D'après le Lemme 3, il existe $X \in \mathfrak{g}$,

$Y \in \mathfrak{g}$ tels que $s' = (\exp iY)(\exp X)$. Alors $y = \sigma'((\exp iY)(\exp X))x = (\exp i\rho(Y))(\exp \rho(X))x$. D'après le Lemme 2 appliqué à y et $(\exp \rho(X))x$, on a $y = (\exp \rho(X))x = \sigma(\exp X)x$. D'où le lemme.

Démonstration du Théorème 1. Soient G un groupe de Lie réel connexe nilpotent, H un sous-groupe abélien fermé distingué connexe, $\hat{H}$ le dual de G. Nous allons montrer que H est régulièrement contenu dans G. Soit K le plus grand sous-groupe compact de H. Il est invariant par tous les automorphismes de H, donc est distingué dans G. Soit L le groupe de recouvrement de H; c'est un espace vectoriel de dimension finie, et H est un quotient de L. Soit $\hat{L}$ le groupe dual de L; il s'identifie à l'espace vectoriel dual de l'espace vectoriel L, et $\hat{H}$ s'identifie à un sous-groupe fermé de $\hat{L}$.

Pour tout $s \in G$, soient $\tau(s)$ l'automorphisme $x \to sxs^{-1}$ de H, $\sigma'(s)$ l'automorphisme correspondant de L, $\sigma(s)$ l'automorphisme de $\hat{L}$ dual de $\sigma'(s)$. Alors, σ' est une représentation linéaire de G dans L qui laisse stable le noyau de l'application canonique de L sur H, donc σ est une représentation linéaire de G dans $\hat{L}$ qui laisse stable $\hat{H}$. Soient $\mathfrak{g}$ l'algèbre de Lie de G, $\mathfrak{h} \subset \mathfrak{g}$ l'algèbre de Lie de H, ou de L. Alors la représentation de $\mathfrak{g}$ correspondant à σ' est la représentation adjointe ρ' de $\mathfrak{g}$ dans $\mathfrak{h}$; elle s'effectue par des endomorphismes nilpotents. La représentation ρ de $\mathfrak{g}$ correspondant à σ est la transposée de ρ'; elle s'effectue également par des endomorphismes nilpotents. On est donc dans les conditions d'application du Lemme 4. Ce lemme entraîne que $\hat{H}$ est réunion de sous-ensembles boréliens deux à deux disjoints $B_1, \cdots, B_n$, stables pour G, possédant les propriétés suivantes: chaque B_i est un espace métrique à base dénombrable, et la relation d'équivalence définie par G dans chaque B_i est ouverte et séparée. Il en résulte aussitôt que H est régulièrement contenu dans G. D'où le Théorème 1.

Remarque. En fait, on déduit facilement de la démonstration que H est même "régulier dans G" au sens de F. Bruhat ("Sur les représentations induites des groupes de Lie," *Bull. Soc. Math. France*, vol. 84 (1956), pp. 97-205, définition 5.3).

2. Démonstration des Théorèmes 2 et 3.

LEMME 5. *Soient G un groupe de Lie réel connexe nilpotent, H un sous-groupe abélien fermé distingué connexe, $\hat{H}$ le dual de H, χ un élément de $\hat{H}$, S le stabilisateur de χ dans G. Alors, S est connexe.*

Démonstration. Nous conservons les notations de la démonstration précédente. Soit $s \in S$, et montrons que s appartient à un sous-groupe à un

paramètre de S. Puisque G est nilpotent, on a $s = \exp X$ avec un $X \in \mathfrak{g}$. Alors, $\sigma(s) = \exp \rho(X)$, et $\rho(X) = \sigma(s) - 1 - \frac{1}{2}(\sigma(s) - 1)^2 + \frac{1}{3}(\sigma(s) - 1)^3 - \cdots$ (les termes de la série étant nuls à partir d'un certain rang). Donc $\rho(X)_\chi = 0$. Alors, $(\sigma(\exp(tX)))\chi = (\exp(t\rho(X)))\chi = \chi$ pour tout nombre réel t, d'où notre assertion.

LEMME 6. *Soit $\mathfrak{g}$ une algèbre de Lie nilpotente de dimension > 1. Tout idéal abélien maximal de $\mathfrak{g}$ est de dimension > 1.*

Démonstration. Soit $\mathfrak{h}$ un idéal de $\mathfrak{g}$ de dimension 1. D'après le théorème d'Engel, il existe une suite d'idéaux $(\mathfrak{h}_0, \mathfrak{h}_1, \cdots, \mathfrak{h}_n)$ de $\mathfrak{g}$ tels que $\mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_n = \mathfrak{g}$, $\dim \mathfrak{h}_i/\mathfrak{h}_{i-1} = 1$, $[\mathfrak{g}, \mathfrak{h}_i] \subset \mathfrak{h}_{i-1}$. Soit $\mathfrak{k}$ un sous-espace de dimension 1 de $\mathfrak{h}_1$ tel que $\mathfrak{h}_1 = \mathfrak{h}_0 + \mathfrak{k}$. On a

$$[\mathfrak{h}_1, \mathfrak{h}_1] = [\mathfrak{h}_1, \mathfrak{h}_0] + [\mathfrak{k}, \mathfrak{k}] \subset [\mathfrak{g}, \mathfrak{h}_0] = 0,$$

donc $\mathfrak{h}_1$ est un idéal abélien contenant strictement $\mathfrak{h}_0$. D'où le lemme.

Démonstration du Théorème 2. Soient G un groupe de Lie réel connexe nilpotent, H un sous-groupe abélien fermé distingué connexe de G, $s \to U_s$ une représentation unitaire factorielle continue de G dans un espace hilbertien $\mathfrak{H}$.

a) Nous supposons d'abord que $\mathfrak{H}$ est à base dénombrable. Soient $\hat{H}$ le groupe dual de H, V la restriction de U à H, M le noyau de V, N la composante connexe de M. Il est clair que M est un sous-groupe fermé distingué de G, donc il en est de même de N.

Le théorème est évident si $H = G$. Soit $n = \dim G - \dim H$. Nous procéderons par récurrence sur n, en supposant le théorème démontré lorsque $\dim G - \dim H < n$.

D'après le Théorème 1, H est régulièrement contenu dans G. Puisque $\hat{H}$ est à base dénombrable, il existe ([6]) un point $\psi \in \hat{H}$ possédant les propriétés suivantes: 1) si O est l'orbite de ψ relativement à G, la mesure sur $\hat{H}$ associée à la décomposition spectrale de V (théorème de Stone généralisé) est concentrée sur O; 2) si S est le stabilisateur de ψ dans G, U est unitairement équivalente à la représentation de G induite par une représentation unitaire factorielle continue W de S. Distinguons alors deux cas.

1) Si $O = \{\psi\}$, on a $H \subset S \neq G$, donc $\dim S - \dim H < \dim G - \dim H$. Comme S est connexe (Lemme 5) l'hypothèse de récurrence prouve qu'il existe un sous-groupe fermé connexe $G' \supset H$ de S et une représentation unitaire continue U' de G' par des opérateurs scalaires tels que W soit unitairement équivalente à la représentation de S induite par U'. Alors, d'après le théorème

sur les représentations induites par étages ([7], th. 4.1), U est unitairement équivalente à la représentation de G induite par U'.

2) Si $O=\{\psi\}$, on a $V(s)=\psi(s)\cdot 1$ pour $s\in H$, donc $\dim N$ est égal à $\dim H$ ou à $(\dim H - 1)$. Supposons d'abord N réduit à l'élément neutre. Alors, H est de dimension 0 ou 1. Si $\dim G = 1$, le théorème est évident. Si $\dim G > 1$, le Lemme 6 prouve qu'il existe un sous-groupe abélien fermé distingué connexe H_1 de G contenant H tel que $\dim G - \dim H_1 < \dim G - \dim H$. Le théorème résulte alors de l'hypothèse de récurrence. Venons-en au cas où N est quelconque. Soient $\tilde{G} = G/N$, $\tilde{H} = H/N$, $\tilde{U}$ la représentation factorielle de $\tilde{G}$ déduite de U par passage au quotient. D'après ce qu'on vient de voir, il existe un sous-groupe fermé connexe $\tilde{G}'$ contenant $\tilde{H}$ de $\tilde{G}$ et une représentation unitaire continue $\tilde{U}'$ de $\tilde{G}'$ par des opérateurs scalaires tels que $\tilde{U}$ soit unitairement équivalente à la représentation de $\tilde{G}$ induite par $\tilde{U}'$. Soient G' l'image réciproque de $\tilde{G}'$ dans G pour l'application canonique θ de G sur $\tilde{G}$, et $U' = \tilde{U}'\cdot\theta$, qui est une représentation unitaire continue de G' par des opérateurs scalaires. Le groupe G' est fermé, contient H, et est connexe parce que N est connexe. Il est immédiat que U est unitairement équivalente à la représentation de G induite par U'. Ceci achève la démonstration lorsque $\mathfrak{H}$ est à base dénombrable.

b) Supposons maintenant que $\mathfrak{H}$ ne soit pas à base dénombrable. Soient x un élément non nul de $\mathfrak{H}$, et (s_j) une suite partout dense dans G. Les $U_{s_j}x$ engendrent un sous-espace vectoriel fermé non nul $\mathfrak{K}$ de $\mathfrak{H}$, stable pour U, et à base dénombrable. Soit A le facteur engendré par les opérateurs U_s, et soit A' le facteur commutant de A. D'après des résultats connus sur les algèbres de von Neumann (cf. par exemple [5], chap. III, § 1, cor. 2 du th. 1), il existe une famille $(\mathfrak{K}_\iota)_{\iota\in I}$ de sous-espaces vectoriels fermés de $\mathfrak{H}$, deux à deux orthogonaux, stables pour A, de somme hilbertienne $\mathfrak{H}$, et tous équivalents à $\mathfrak{K}$ relativement à A'. Soit W la représentation de G obtenue en restreignant à $\mathfrak{K}$ les opérateurs U_s. Alors, U est somme hilbertienne d'une famille $(W_\iota)_{\iota\in I}$ de représentations unitairement équivalentes à W. D'après la partie a) de la démonstration, il existe un sous-groupe fermé connexe $G'\supset H$ de G, et une représentation unitaire continue W' de G' par des opérateurs scalaires, tels que W soit unitairement équivalente à la représentation de G induite par W'. Pour tout $\iota\in I$, soit W'_ι une représentation de G' unitairement équivalente à W', et U' la somme hilbertienne des W'_ι. Alors ([7], th. 10.1), U est unitairement équivalente à la représentation de G induite par U'.

Démonstration du Théorème 3. Soient G un groupe de Lie réel connexe nilpotent, $s \to U_s$ une représentation unitaire continue de G dans un espace hilbertien $\mathfrak{H}$.

a) Supposons U factorielle et $\mathfrak{H}$ à base dénombrable. Soit n un entier > 0. Le théorème est évident pour les groupes abéliens. Supposons le théorème établi pour les groupes de Lie réels connexes nilpotents G possédant un sous-groupe abélien fermé distingué connexe H tel que $\dim G - \dim H < n$. Considérons alors le cas où G possède un sous-groupe abélien fermé distingué connexe H tel que $\dim G - \dim H = n$.

La marche de la démonstration est alors la même que pour le Théorème 2. Soient V la restriction de U à H, O l'orbite dans $\hat{H}$ associée à V, ψ un point de O, S le stabilisateur de ψ dans G. On sait que U est unitairement équivalente à la représentation de G induite par une représentation unitaire continue U' de S. En outre, les projecteurs du système d'imprimitivité associé à cette représentation induite proviennent de la décomposition spectrale de V, donc appartiennent à l'algèbre de von Neumann engendrée par les U_s, $s \in G$. D'après [6], no. 6, l'algèbre de von Neumann B formée des opérateurs permutables aux U'_s, $s \in S$, est donc isomorphe à l'algèbre de von Neuman A formée des opérateurs permutables aux U_s, $s \in G$.

1) Si $O \neq \{\psi\}$, on a $H \subset S \neq G$, donc $\dim S - \dim H < \dim G - \dim H$. Comme S est connexe (Lemme 5), l'hypothèse de récurrence prouve que U' est de type I. Donc B est de type I, donc A est de type I, donc U est de type I.

2) Si $O = \{\psi\}$, on a $V(s) = \psi(s) \cdot 1$ pour $s \in H$. Soit M le noyau de V, qui est un sous-groupe fermé distingué de G. Soient $\tilde{G} = G/M$, $\tilde{H} = H/M$. Alors, $\tilde{H}$ est de dimension 0 ou 1. Si $\tilde{G}$ est de dimension 1, les U_s ($s \in G$) sont deux à deux permutables, et le théorème est évident. Si $\dim \tilde{G} > 1$, il existe (Lemme 6) un sous-groupe abélien fermé distingué connexe $\tilde{H}'$ de $\tilde{G}$, contenant $\tilde{H}$, et de dimension > 1. D'après l'hypothèse de récurrence, la représentation $\tilde{U}$ de G déduite de U par passage au quotient est de type I. Donc U elle-même est de type I.

b) Supposons toujours $\mathfrak{H}$ à base dénombrable, mais U quelconque. Soit $U_s = \int^{\oplus} U^y{}_s \, d\mu(y)$ la décomposition de U en représentations factorielles $s \to U^y{}_s$. D'après la partie a) de la démonstration, les représentations U^y sont de type I. Donc ([8], th. 2.6) U est de type I.

c) Envisageons enfin le cas général. Soit $(\mathfrak{H}_\iota)_{\iota \in I}$ une famille maximale

de sous-espaces vectoriels fermés non nuls de $\mathfrak{H}$, deux à deux orthogonaux, stables pour U, tels que les représentations U_i obtenues en restreignant U aux $\mathfrak{H}_i$ soient de type I. On va montrer que $\mathfrak{H} = \bigoplus_{i \in I} \mathfrak{H}_i$. Raisonnant par l'absurde, supposons qu'il existe un élément $x \neq 0$ de $\mathfrak{H}$ orthogonal aux $\mathfrak{H}_i$. Soit (s_j) une suite partout dense dans G. Les $U_{s_j}x$ engendrent un sous-espace vectoriel fermé non nul $\mathfrak{K}$ de $\mathfrak{H}$, stable pour U, orthogonal aux $\mathfrak{H}_i$, et à base dénombrable. D'après la partie b) de la démonstration, la représentation obtenue en restreignant U à $\mathfrak{K}$ est de type I. Ceci contredit la maximalité de la famille $(\mathfrak{H}_i)_{i \in I}$. Donc $\mathfrak{H} = \bigoplus_{i \in I} \mathfrak{H}_i$. Il est alors bien connu que U est de type I (cf. par exemple [5], chap. III, § 2, exerc. 5a)). Ceci achève la démonstration.

Corollaire du Théorème 3. *Toute représentation unitaire projective continue d'un groupe de Lie réel connexe nilpotent G est de type I.*

Démonstration. Soit U une telle représentation. On peut supposer G simplement connexe. D'après [1], U provient par passage au quotient d'une représentation unitaire continue d'un groupe de Lie réel connexe G', extension centrale de G. Alors, G' est nilpotent, donc U est de type I.

3. Un contre-exemple. Nous allons construire un groupe de Lie réel connexe nilpotent G et un sous-groupe fermé abélien *non* connexe H de G, tels que H ne soit pas régulièrement contenu dans G.

Soient $\boldsymbol{R}$ le groupe additif des nombres réels, $\boldsymbol{Z}$ le sous-groupe des entiers rationnels, et $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$. Définissons sur $\boldsymbol{R}^4$ une multiplication de la manière suivante:

$$(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 - x_2 y_1 - 2^{\frac{1}{2}} x_3 y_1).$$

On vérifie aisément que cette multiplication est associative, que $(0,0,0,0)$ est élément neutre, et que $(-x_1, -x_2, -x_3, -x_4 - x_1 x_2 - 2^{\frac{1}{2}} x_1 x_3)$ est inverse de (x_1, x_2, x_3, x_4). On a donc défini un groupe de Lie réel G, connexe. Les éléments dont les trois premières coordonnées sont nulles forment un sous-groupe C. On a la formule

$$(1)\quad (x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4)(x_1, x_2, x_3, x_4)^{-1} = (y_1, y_2, y_3, y_4 + x_1 y_2 - x_2 y_1 + 2^{\frac{1}{2}}(x_1 y_3 - x_3 y_1)).$$

Sur la formule (1), on voit que C est central; comme G/C est évidemment abélien, G est nilpotent.

Soit H l'ensemble des $(x_1, x_2, x_3, x_4) \in G$ tels que x_1 soit nul et x_2, x_3 entiers. Alors, H est un sous-groupe fermé distingué de G, abélien puisque

$$(0, x_2, x_3, x_4)(0, y_2, y_3, y_4) = (0, x_2 + y_2, x_3 + y_3, x_4 + y_4).$$

Soient $x = (0, y_2, y_3, y_4) \in H$, $s = (x_1, x_2, x_3, x_4) \in G$, et $sxs^{-1} = (0, y_2', y_3', y_4')$ $\in H$. La formule (1) prouve que

$$y_2' = y_2 \qquad y_3' = y_3 \qquad y_4' = y_4 + x_1 y_2 + 2^{\frac{1}{2}} x_1 y_3.$$

Le dual $\hat{H}$ de H s'identifie à $\boldsymbol{T} \times \boldsymbol{T} \times \boldsymbol{R}$. Si $(\theta_2, \theta_3, \xi_4) \in \hat{H}$ (où $\theta_2 \in \boldsymbol{T}, \theta_3 \in \boldsymbol{T}$, $\xi_4 \in \boldsymbol{R}$), son transformé par s est $(\theta_2', \theta_3', \xi')$, avec $\theta_2' = \theta_2 + x_1 \xi_4$, $\theta_3' = \theta_3$ $+ 2^{\frac{1}{2}} x_1 \xi_4$, $\xi_4' = \xi_4$.

Dans chaque sous-ensemble A_ξ de $\hat{H}$ défini par une valeur ξ de ξ_4, le groupe G agit ergodiquement pour la mesure de Haar de $\boldsymbol{T}^2$ (à cause de l'irrationnalité de $2^{\frac{1}{2}}$). Les parties de $\hat{H}$ mesurables pour la mesure de Haar de $\hat{H}$ et stables pour G sont donc, à des ensembles négligeables près, des réunions d'ensembles A_ξ. Ainsi, H n'est pas régulièrement contenu dans G.

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GEOMETRIC SYZYGIES.* †

By G. Washnitzer.

The explanation of the title is that we give here a systematic interpretation of an exact sequence of sheaves in terms of geometric entities. In particular, we examine from this standpoint Hilbert's theorem concerning "chains of syzygies," and this provides an axiomatic characterization of the arithmetic genus of a non-singular projective model.

Given a coherent, algebraic, locally free sheaf E defined on a variety X, we attach a projective fiber bundle $\mathcal{B}(E)$, the "dual projective bundle" of E, whose base space is X. The fiber of $\mathcal{B}(E)$ is a projective space of dimension $n-1$, where n is equal to the dimension of the fiber of the vector bundle whose sheaf of germs of (algebraic) cross-sections is the sheaf E; in fact, the fiber of $\mathcal{B}(E)$ is equal to projective space formed by the vector subspaces of dimension $n-1$ in the fiber of the vector bundle. Let π_E denote the bundle projection of $\mathcal{B}(E)$ and let $\pi_E E$ denote the reciprocal image sheaf of E with respect to π_E, so that $\pi_E E$ is a locally free sheaf of dimension n defined on $\mathcal{B}(E)$. We construct a locally free sheaf $B(E)$ of dimension one defined on $\mathcal{B}(E)$ and a homomorphism from $\pi_E E$ onto $B(E)$. The kernel of this homomorphism is a locally free sheaf δE of dimension $n-1$ defined on $\mathcal{B}(E)$ and we have the following exact sequence of locally free sheaves defined on $\mathcal{B}(E)$:

$$0 \to \delta E \to \pi_E E \to B(E) \to 0.$$

$B(E)$ (resp. δE) is called the "basic sheaf" (resp. "derived sheaf") of the sheaf E. Assuming that X is non-singular (this is always the assumption in the text), we have that $B(E)$ is necessarily isomorphic with the sheaf of germs of rational functions on $\mathcal{B}(E)$ which are multiples of some divisor ϑ on $\mathcal{B}(E)$. The divisor class $\Theta(E)$ of ϑ depends solely upon the sheaf E, and $\Theta(E)$ is called the "basic class" of E.

Let E, G be locally free sheaves defined on X, and let ψ be a homomorphism from G into E with the property that the image sheaf $\mathrm{Im}[\psi\colon G \to E]$ is not the zero sheaf. Then we construct a rational transformation $\mathcal{B}(\psi)$

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from $\mathcal{B}(E)$ to $\mathcal{B}(G)$; $\mathcal{B}(\psi)$ is called the "dual rational transformation" of the homomorphism ψ. Let $\mathcal{G}(\psi)$ denote the graph of $\mathcal{B}(\psi)$ and let $\psi_{;1}$, $\psi_{;2}$ denote the projections from $\mathcal{G}(\psi)$ onto $\mathcal{B}(E)$, $\mathcal{B}(G)$ respectively. We pass to sheaves $\psi_{;1}B(E)$, $\psi_{;2}B(G)$, the reciprocal images of $B(E)$ and $B(G)$ with respect to $\psi_{;1}$ and $\psi_{;2}$ respectively, which are both locally free sheaves of dimension one defined on $\mathcal{G}(\psi)$. We then construct a locally free sheaf $S(\psi)$ of dimension one defined on $\mathcal{G}(\psi)$ and an isomorphism of the tensor product sheaf $\psi_{;2}B(G) \otimes S(\psi)$ onto $\psi_{;1}B(E)$.

A grave difficulty now arises. For, it can occur, in the absence of any further hypotheses, that the variety $\mathcal{G}(\psi)$ has multiple points; we can assume that X is non-singular. Furthermore, it can, and does indeed, occur that $\mathcal{G}(\psi)$ is non-singular, but that its projection into $\mathcal{B}(G)$ has multiple points. These possibilities present a serious obstacle to any detailed examination of the rational transformation $\mathcal{B}(\psi)$. Fortunately, there are some special cases which can be analyzed completely and which are of sufficient scope to cover a wide range of application.

Consider the case of an exact sequence

$$0 \to H \xrightarrow{\theta} G \xrightarrow{\psi} E \to 0$$

of locally free sheaves defined on X. In this case, $\mathcal{B}(\psi)$ is a bi-regular mapping from $\mathcal{B}(E)$ onto a subvariety on $\mathcal{B}(G)$. Identifying $\mathcal{B}(E)$ with that subvariety, we have that the restriction of π_G to $\mathcal{B}(E)$ is equal to π_E, and that the trace of the divisor class $\Theta(G)$ on $\mathcal{B}(E)$ is equal to $\Theta(E)$. The graph $\mathcal{G}(\theta)$ is a non-singular variety; it is obtained by performing monoidal transformation on $\mathcal{B}(G)$ with the subvariety $\mathcal{B}(E)$ for center and the projection $\theta_{;1}$ is the anti-monoidal transformation. The projection $\theta_{;2}$ from $\mathcal{G}(\theta)$ to $\mathcal{B}(H)$ equips $\mathcal{G}(\theta)$ with the structure of the dual projective bundle of a certain locally free sheaf defined on $\mathcal{B}(H)$. In the present situation, we have that

$$\Theta(\theta_{;2}B(H)) + \Theta(S(\theta)) = \Theta(\theta_{;1}B(G)),$$

where the basic class $\Theta(S(\theta))$ of $S(\theta)$ is the divisor class of the anti-center $\mathfrak{N}_\theta$ of $\theta_{;1}$; $\mathfrak{N}_\theta$ is, of course, a non-singular subvariety of co-dimension one on $\mathcal{G}(\theta)$.

Next, let ψ be a homomorphism from G into E with the following property: The residue class sheaf Q of E modulo $\operatorname{Im}[\psi\colon G \to E]$ has for support a non-singular proper subvariety V on X, and Q is the extension to

X of a locally free sheaf defined on the subvariety V; thus we have the exact sequence

$$G \xrightarrow{\psi} E \xrightarrow{\phi} Q \to 0,$$

where ϕ is the quotient mapping from E to Q. Now $\mathbf{B}(Q)$ is a projective fiber bundle whose base space is V. In the first instance, $\mathbf{B}(\phi)$ makes no sense since Q is not a locally free sheaf on X; but we can define, in a natural way, a bi-regular mapping $\mathbf{B}(\phi)$ of $\mathbf{B}(Q)$ onto a non-singular subvariety on $\mathbf{B}(E)$. Identifying $\mathbf{B}(Q)$ with that subvariety, we have that the restriction of π_E to $\mathbf{B}(Q)$ is equal to π_Q and that the trace of $\Theta(E)$ on $\mathbf{B}(Q)$ is equal to $\Theta(Q)$. The graph $\mathbf{G}(\psi)$ is obtained by performing monoidal transformation on $\mathbf{B}(E)$ with the subvariety $\mathbf{B}(Q)$ for center; hence, $\mathbf{G}(\psi)$ is a non-singular variety and $\psi_{;1}$ is the anti-monoidal transformation. On the other hand, the subvariety on $\mathbf{B}(G)$ obtained by applying the projection $\psi_{;2}$ to $\mathbf{G}(\psi)$ has multiple points except for the following cases.

(a) V is a subvariety of co-dimension one on X.

(b) The dimension of the locally free sheaf Q on V is equal to the dimension of the locally free sheaf E on V.

Consider the case (a). Since V is of co-dimension one the kernel of ψ is a locally free sheaf defined on X, so that there is no loss of generality if we confine our attention to an exact sequence

$$0 \to G \xrightarrow{\psi} E \xrightarrow{\phi} Q \to 0$$

—E, G, locally free sheaves on X, Q is the extension of a locally free sheaf defined on V, V is a non-singular subvariety of co-dimension one on X. In this case, we prove that $\psi_{;2}$ is an anti-monoidal transformation from $\mathbf{G}(\psi)$ onto $\mathbf{B}(G)$; the center is a non-singular subvariety on $\mathbf{B}(G)$. Again we have that

$$\Theta(\psi_{;2}B(G)) + \Theta(S(\psi)) = \Theta(\psi_{;1}B(E)),$$

where $\Theta(S(\psi))$ is the divisor class of $\mathfrak{N}_\psi$; $\mathfrak{N}_\psi$ is a non-singular subvariety of co-dimension one on $\mathbf{G}(\psi)$ and it is the anti-center of $\psi_{;1}$. There is a detailed description of $\psi_{;2}$ in the text (§ 19).

Consider the following exact sequence of sheaves defined on a non-singular variety X:

$$F_t \to \cdots \to F_s \xrightarrow{\psi_s} F_{s-1} \to \cdots \to F_0 \xrightarrow{\psi_0} Q \to 0,$$

where $F_0, \cdots, F_t$ are locally free sheaves defined on X and Q is the extension to X of a locally free sheaf defined on V; V is a non-singular proper subvariety on X. Set $r = \dim. X$, $d = \dim. V$. Then, from the local version of Hilbert's syzygies theorem, it is well known that $\operatorname{Ker}[\psi_s : F_s \to F_{s-1}]$ is a locally free sheaf on X if $s \geqq r - d - 1$. Consequently, there exist exact sequences

$$0 \to F_{r-d} \xrightarrow{\psi_{r-d}} \cdots \to F_s \xrightarrow{\psi_s} F_{s-1} \to \cdots \xrightarrow{\psi_0} Q \to 0,$$

where $F_0, \cdots, F_{r-d}$ are locally free; such exact sequences are called "resolutions" of Q by locally free sheaves. The foregoing discusssion permits the introduction of a "geometric resolution." One replaces each sheaf by its dual projective bundle, and each homomorphism by its dual rational transformation. However, we must use the following device so as to avoid the intractable cases where the "graphs" $\mathcal{G}(\psi_s)$ admit multiple points. Let X^* be obtained by performing monoidal transformation to X centered on V, let Φ denote the anti-monoidal transformation from X^* onto X, and V^* denote the anti-center, so that V^* is a non-singular subvariety of dimension $r-1$ on X^*. We set

$$F_s^* = \Phi F_s,$$

$$Q^* = \Phi Q,$$

so that F_s^* (resp. Q^*) is a locally free sheaf defined on X^* (resp. V^*), and we let

$$\psi_s^* : F_s^* \to F_{s-1}^*, \qquad 1 \leqq s \leqq r-d,$$

$$\psi_0^* : F_0^* \to Q^*,$$

where ψ_s^* is the reciprocal image of ψ_s with respect to Φ (see §§ 1, 2). The diagram of sheaves and homomorphisms on X^*

$$0 \to F_{r-d}^* \xrightarrow{\psi_{r-d}^*} \cdots \to F_s^* \xrightarrow{\psi_s^*} F_{s-1}^* \to \cdots \to F_0^* \to Q^* \to 0$$

is not exact if $d > 1$; but for $d > 1$, the following is true:

$$\operatorname{Ker}[\psi_s^*], \qquad 0 \leqq s \leqq r-d-1,$$

$$\operatorname{Im}[\psi_s^*], \qquad 1 \leqq s \leqq r-d,$$

are locally free sheaves on X^*; we have the exact sequences

$$0 \to \operatorname{Ker}[\psi_s^*] \to F_s^* \to \operatorname{Im}[\psi_s^*] \to 0, \qquad 1 \leqq s \leqq r-d-1,$$

of locally free sheaves defined on X^*; and the exact sequences

$$0 \to \mathrm{Im}[\psi_1{}^*] \to F_0{}^* \to Q^*$$

$$0 \to \mathrm{Im}[\psi_{s+1}{}^*] \to \mathrm{Ker}[\psi_s{}^*] \to Q^* \otimes \wedge^s(\delta N) \to 0, \qquad 1 \leqq s \leqq r-d-1.$$

The sheaf δN is the derived sheaf of the locally free sheaf $N(X;V)$ of dimension $r-d$ on V of germs of cross-sections of covariant normal vector fields to V (see § 9); $\wedge^s(\delta N)$ is the s-fold exterion product of δN, and we recall that δN is a locally free sheaf of dimension $r-d-1$ defined on V^*.

Chapter IV is an application of the method of "geometric resolution." We operate with an axiomatically defined "arithmetic functional" $\mathfrak{a}$ which assigns a rational number $\mathfrak{a}(X)$ to each non-singular projective model X. The decisive axiom (§ 21) is the Fiber Law:

(a) If Y is the dual projective bundle of a locally free sheaf defined on a non-singular projective model X. Then $\mathfrak{a}(X) = \mathfrak{a}(Y)$.

(b) If Y is obtained by performing monoidal transformation of a non-singular projective model X with center on a non-singular subvariety V, then $\mathfrak{a}(Y) = \mathfrak{a}(X)$. We proceed to prove that $\mathfrak{a}$ is unique by showing that $\mathfrak{a}(X) = \chi(X, \mathcal{O}_X)$, the Euler-Poincaré characteristic of the sheaf $\mathcal{O}_X$ of local rings on X. The equality $\mathfrak{a}(X) = \chi(X, \mathcal{O}_X)$ is proved merely on the assumption that $\mathfrak{a}$ exists, and we are not required to know that χ satisfies the Fiber Law. Indeed, χ does satisfy the Fiber Law; this being well known for non-singular varieties defined over a field of characteristic zero, and it has been proved by J. H. Sampson and the author for arbitrary characteristic.

We prefer, however, to establish the existence of an arithmetic functional along other lines. In a subsequent publication, we shall demonstrate that the Todd genus $T(X)$ satisfies the axioms; hence, we obtain

$$T(X) = \chi(X, \mathcal{O}_X)$$

which is the Todd-Hirzebruch formula for (non-singular) varieties defined over fields of arbitrary characteristic.

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The principal references for the investigations of the present paper are the following articles:

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I. The Dual Projective Bundle and the Basic Sheaf of a Locally Free Sheaf.

§ 0. The term "variety" will refer always to an irreducible algebraic variety defined over a fixed algebraically closed field K of arbitrary characteristic. Our varieties are all equipped with the Zariski topology (with reference to K of course), and whenever we say that a variety is a subvariety on some other variety, we shall understand that the former is a closed subset on the latter. Every variety under consideration here will be either (a) a projective model (i. e., a subvariety on some projective space), or (b) it will admit a bi-regular correspondence onto an open subset of some projective model. There is, however, one instance where we construct a variety which a priori is merely an abstract variety; but we prove that this variety admits a bi-regular correspondence onto some projective model.

The term "sheaf" will always refer to a coherent algebraic sheaf defined on some variety, and a homomorphism of such sheaves will always be an algebraic homomorphism.

§ 1. U, V are varieties; Φ is a rational transformation from V to U. The graph of Φ is a subvariety Z on the product variety $V \times U$. We assume that Φ is regular. This means that for every point $\mathfrak{q}$ on V, there is one and only one point $\mathfrak{z} = (\mathfrak{q}, \mathfrak{p})$ on Z which projects onto $\mathfrak{q}$, and that the induced homomorphism of the local ring $\mathfrak{O}_{\mathfrak{q}}$ of V at $\mathfrak{q}$ into the local ring $\mathfrak{O}_{\mathfrak{z}}$ of Z at $\mathfrak{z}$ is an isomorphism of the former ring onto the latter. Let $\mathfrak{p}$ be the

point of U which is the image of $\mathfrak{s}$ by the projection from Z to U. Then we have that Φ induces a homomorphism $\Phi_{\mathfrak{q}}$ of the local ring $\mathcal{O}_{\mathfrak{p}}$ of U at $\mathfrak{p}$ into $\mathcal{O}_{\mathfrak{q}}$, and we set $\mathfrak{p}=\Phi(\mathfrak{q})$. Φ, as mapping of topological spaces, is a continuous map from V to U. We recall that the set theoretic image of V in U by Φ is not necessarily a closed subset on U.

We equip $\mathcal{O}_{\mathfrak{q}}$ with the structure of an $\mathcal{O}_{\mathfrak{p}}$-module according to the rule

$$u \cdot v = \Phi_{\mathfrak{q}}(u) v, \qquad u \in \mathcal{O}_{\mathfrak{p}},\ v \in \mathcal{O}_{\mathfrak{q}},$$

where the right hand side is a product in the ring $\mathcal{O}_{\mathfrak{q}}$. Now let A be a (coherent, algebraic) sheaf defined on U, and let $A_{\mathfrak{p}}$ be the stalk of A at $\mathfrak{p}=\Phi(\mathfrak{q})$. We form the $\mathcal{O}_{\mathfrak{p}}$-tensor product

$$\mathcal{O}_{\mathfrak{q}} \otimes_{\mathcal{O}_{\mathfrak{p}}} A_{\mathfrak{p}};$$

this being permissible since we have equipped $\mathcal{O}_{\mathfrak{q}}$ with the structure of an $\mathcal{O}_{\mathfrak{p}}$-module in a specific way. For notational convenience, let us set

$$A^*_{\mathfrak{q}} = \mathcal{O}_{\mathfrak{q}} \otimes_{\mathcal{O}_{\mathfrak{p}}} A_{\mathfrak{p}}.$$

We have a canonical $\mathcal{O}_{\mathfrak{p}}$-homorphism $\Phi^*_{\mathfrak{q}}$ of $A_{\mathfrak{p}}$ into the $\mathcal{O}_{\mathfrak{p}}$-module $A^*_{\mathfrak{q}}$ defined according to

$$\Phi^*_{\mathfrak{q}}(ua) = 1 \otimes (ua) = \Phi_{\mathfrak{q}}(u) \otimes a, \qquad u \in \mathcal{O}_{\mathfrak{p}}, a \in A_{\mathfrak{p}}.$$

On the other hand, $A^*_{\mathfrak{q}}$ carries the structure of an $\mathcal{O}_{\mathfrak{q}}$-module according to the rule

$$v_1(v_2 \otimes a) = (v_1 v_2) \otimes a, \qquad v_1, v_2 \in \mathcal{O}_{\mathfrak{q}}, a \in A_{\mathfrak{p}}.$$

The $\mathcal{O}_{\mathfrak{p}}$-homomorphism $\Phi^*_{\mathfrak{q}}$ maps any set of generators of $A_{\mathfrak{p}}$ onto a set of generators for $A^*_{\mathfrak{q}}$ viewed as $\mathcal{O}_{\mathfrak{q}}$-module. If $A_{\mathfrak{p}}$ is a free $\mathcal{O}_{\mathfrak{p}}$-module of dimension n, then $A^*_{\mathfrak{q}}$ is a free $\mathcal{O}_{\mathfrak{q}}$-module of dimension n.

§ 2. Set

$$A^* = \bigcup_{\mathfrak{q} \in V} A^*,$$

the disjoint union of the various modules $A^*_{\mathfrak{q}}$ for all $\mathfrak{q} \in V$. We shall prove that the set A^* carries the structure of one and only one coherent algebraic sheaf ΦA defined on V with the following properties:

(a) The stalk of ΦA at any point $\mathfrak{q}$ is the module $A^*_{\mathfrak{q}}$;

(b) Let γ be a section of A over an open set W on U, then the mapping $\Phi^{-1}\gamma$ which assigns to each point $\mathfrak{q}$ on $\Phi^{-1}(W)$ ($\Phi^{-1}(W)$ is the open set on V consisting of all points $\mathfrak{q}$ such that $\Phi(\mathfrak{q})$ is in W, and we assume that $\Phi^{-1}(W)$

is non-empty) the element $\Phi^*_{\mathfrak{q}}(\gamma(\mathfrak{p}))$ in $A^*_{\mathfrak{q}}$ ($\gamma(\mathfrak{p})$) is the value of γ in $A_{\mathfrak{p}}$, where $\mathfrak{p} = \Phi(\mathfrak{q})$, and $\Phi^*_{\mathfrak{q}}$ is the previous canonical homomorphism of $A_{\mathfrak{p}}$ into $A^*_{\mathfrak{q}}$) is a section of ΦA over $\Phi^{-1}(W)$. The sheaf ΦA is called the reciprocal image of A with respect to the mapping Φ.

It is clear that any two sheaves defined on V which satisfy both (a) and (b) must be identical. It remains for us to construct the sheaf ΦA. First, we observe that the reciprocal image of the sheaf $\mathcal{O}_U$ of local rings on U is the sheaf $\mathcal{O}_V$ of local rings on V. More generally, if A is a free sheaf of dimension n on U (i.e., A is isomorphic with the direct sum of $\mathcal{O}_U$ taken n times), then ΦA exists and it is a free sheaf of dimension n defined on V. Next, if A is such that we can choose free sheaves B, C defined on U and homomorphisms $\theta\colon C \to B$, $\psi\colon B \to A$ with the property that

$$C \xrightarrow{\theta} B \xrightarrow{\psi} A \to 0$$

is an exact sequence of sheaves defined on U, then ΦA exists. For in the present situation, we have the exact sequence

$$C_{\mathfrak{p}} \xrightarrow{\theta_{\mathfrak{p}}} B_{\mathfrak{p}} \xrightarrow{\psi_{\mathfrak{p}}} A_{\mathfrak{p}} \to 0$$

of stalks at any point $\mathfrak{p} = \Phi(\mathfrak{q})$ on U. Tensorize this exact sequence with $\mathcal{O}_{\mathfrak{q}}$ viewed as $\mathcal{O}_{\mathfrak{p}}$-module, thereby obtaining the exact sequence

$$C^*_{\mathfrak{q}} \xrightarrow{\theta^*_{\mathfrak{q}}} B^*_{\mathfrak{q}} \xrightarrow{\psi^*_{\mathfrak{q}}} A^*_{\mathfrak{q}} \to 0$$

of $\mathcal{O}_{\mathfrak{p}}$-modules; $B^*_{\mathfrak{q}} = \mathcal{O}_{\mathfrak{q}} \otimes_{\mathcal{O}_{\mathfrak{p}}} B_{\mathfrak{p}}$, $C^*_{\mathfrak{q}} = \mathcal{O}_{\mathfrak{q}} \otimes_{\mathcal{O}_{\mathfrak{p}}} C_{\mathfrak{p}}$ and $\theta^*_{\mathfrak{q}} = I_{\mathfrak{q}} \otimes \theta_{\mathfrak{p}}$, $\psi^*_{\mathfrak{q}} = I_{\mathfrak{q}} \otimes \psi_{\mathfrak{p}}$, where $I_{\mathfrak{q}}$ is the identity map of $\mathcal{O}_{\mathfrak{q}}$. We invoke here the well known fact that $\otimes_{\mathcal{O}_{\mathfrak{p}}}$ is a right exact functor. But this last exact sequence is clearly an exact sequence of $\mathcal{O}_{\mathfrak{q}}$-modules since $\theta^*_{\mathfrak{q}}$ and $\psi^*_{\mathfrak{q}}$ are $\mathcal{O}_{\mathfrak{q}}$-homomorphisms. The family of homomorphisms $\{\theta^*_{\mathfrak{q}}\}_{\mathfrak{q} \in V}$ defines an algebraic homomorphism $\Phi^{-1}(\theta)$ of ΦC into ΦB. Consider the quotient sheaf of ΦB modulo the image of ΦC by $\Phi^{-1}(\theta)$. The underlying set of this quotient sheaf can be indentified with A^* in a specific way, and performing this identification, we obtain the sheaf ΦA.

Finally, let A be an arbitrary coherent algebraic sheaf defined on U. Let W be an open set on U such that

1) $\Phi^{-1}(W)$ is non-empty,

2) There exist free sheaves B, C defined on W and homomorphisms $\theta: C \to B$, $\psi: B \to A$ such that

$$C \xrightarrow{\theta} B \xrightarrow{\psi} A \to 0$$

is an exact sequence. Let $A^*(W)$ be the subset of A^* consisting of the union of all $A^*_{\mathfrak{q}}$ with $\mathfrak{q} \in \Phi^{-1}(W)$. Then, by the previous discussion, $A^*(W)$ carries the structure of a coherent algebraic sheaf defined on $\Phi^{-1}(W)$, and this sheaf is the reciprocal image of the restriction of A to W with respect to the restriction of Φ to $\Phi^{-1}(W)$. If W' is another open set on U satisfying 1) and 2), then the restriction of the sheaf $A^*(W)$ to $\Phi^{-1}(W \cap W')$ is identical with the restriction of $A^*(W')$ to $\Phi^{-1}(W \cap W')$; for each of these last sheaves is the reciprocal image of the restriction of A to $W \cap W'$ by the restriction of Φ to $\Phi^{-1}(W \cap W')$. Since we can cover U by a family $\{W\}$ of open sets which satisfy 1) and 2), we obtain the sheaf ΦA defined on V.

Given an open set W on U, with $\Phi^{-1}(W)$ non-empty, there is a canonical homomorphism of the module $\Gamma(A, W)$ of sections of A over W into the module $\Gamma(\Phi A, \Phi^{-1}(W))$ of sections of ΦA over $\Phi^{-1}(W)$; this follows from condition (b) in the definition of ΦA. The module $\Gamma(\mathcal{O}_U, W)$ (resp. $\Gamma(\mathcal{O}_V, \Phi^{-1}(W)))$ is the same as the ring of regular functions on W (resp. $\Phi^{-1}(W)$), and the canonical homomorphism of $\Gamma(\mathcal{O}_U, W)$ into $\Gamma(\mathcal{O}_V, \Phi^{-1}(W))$ is a ring homomorphism of the former ring into the latter. This permits us to view $\Gamma(\mathcal{O}_V, \Phi^{-1}(W))$ as module over $\Gamma(\mathcal{O}_U, W)$. More generally, $\Gamma(\Phi A, \Phi^{-1}(W))$ carries the structure of a $\Gamma(\mathcal{O}_U, W)$-module, and the canonical homomorphism of $\Gamma(A, W)$ into $\Gamma(\Phi A, \Phi^{-1}(W))$ is a $\Gamma(\mathcal{O}_U, W)$-homomorphism.

A, B are sheaves defined on U, and ψ is a homomorphism of B into A. We construct an $\mathcal{O}_{\mathfrak{q}}$-homomorphism $\psi^*_{\mathfrak{q}}$ of $B^*_{\mathfrak{q}}$ (the stalk of ΦB at $\mathfrak{q}$) into $A^*_{\mathfrak{q}}$ (the stalk of ΦA at $\mathfrak{q}$) according to the rule

$$\psi^*_{\mathfrak{q}}: v \otimes b \to v \otimes \psi_{\mathfrak{p}}(b), \qquad v \in \mathcal{O}_{\mathfrak{q}}, b \in B_{\mathfrak{p}},$$

where $\psi_{\mathfrak{p}}$ is the homomorphism of $B_{\mathfrak{p}}$ into $A_{\mathfrak{p}}$ assigned by ψ. One readily proves that the family of homomorphisms $\{\psi^*_{\mathfrak{q}}\}_{\mathfrak{q} \in V}$ gives a homomorphism $\Phi^{-1}(\psi)$ of ΦB into ΦA, which we call the reciprocal image of the homomorphism ψ. For any open set W on U, with $\Phi^{-1}(W)$ non-empty, we have the following commutative diagram.

$$\begin{array}{ccc}
\Gamma(B, W) & \longrightarrow & \Gamma(A, W) \\
\downarrow & & \downarrow \\
\Gamma(\Phi B, \Phi^{-1}(W)) & \to & \Gamma(\Phi A, \Phi^{-1}(W));
\end{array}$$

the horizontal arrows are induced by the sheaf homomorphisms ψ and $\Phi^{-1}(\psi)$; the vertical arrows are the aforementioned canonical homomorphisms.

The chief peculiarity with the notion of the reciprocal image of an algebraic sheaf is that exact sequences are not generally preserved. Given an exact sequence

$$0 \to C \xrightarrow{\theta} B \xrightarrow{\psi} A \to 0$$

of sheaves defined on U, then we have the exact sequence

$$\Phi C \xrightarrow{\Phi^{-1}(\theta)} \Phi B \xrightarrow{\Phi^{-1}(\psi)} \Phi A \to 0$$

of sheaves defined on V; but it is not always true that the kernel of $\Phi^{-1}(\theta)$ is the sheaf zero. Further on, we shall encounter some interesting examples where exactness fails to be preserved.

In the situation where V is a subvariety on U and Φ is the identity map of V in U, we shall call ΦA the induced sheaf of A on the subvariety V.

§3. X is a non-singular projective model, and E is a locally free sheaf of dimension n $(n > 0)$ defined on X (i.e., the stalk $E_{\mathfrak{p}}$ of E at any point $\mathfrak{p}$ on X is a free $\mathcal{O}_{\mathfrak{p}}$-module of dimension n). It is possible to choose an arbitrarily fine covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in J}$ of X consisting of finitely many non-empty open sets U_α on X such that:

1) Each U_α admits a bi-regular correspondence onto some affine model (i.e., a subvariety on some affine space),

2) The restriction of E to any U_α is a free sheaf of dimension n. Consequently, we can choose for each $\alpha \in J$ sections $e^\alpha{}_1, \cdots, e^\alpha{}_n$ of E over U_α such that for each $\mathfrak{p} \in U_\alpha$, the elements $e^\alpha{}_1(\mathfrak{p}), \cdots, e^\alpha{}_n(\mathfrak{p})$ generate $E_{\mathfrak{p}}$ as free $\mathcal{O}_{\mathfrak{p}}$-module of dimension n. For an ordered pair (α, β), we have that

$$e^\alpha{}_j = \sum_{i=1}^{n} E^{\beta}{}_i{}^{\alpha}{}_j e^{\beta}{}_i, \qquad 1 \leqq j \leqq n,$$

where the $E^{\beta}{}_i{}^{\alpha}{}_j$ are rational functions on X which are regular at each point of the set theoretic intersection $U_\alpha \cap U_\beta$. Thus for each point $\mathfrak{p}$ on $U_\alpha \cap U_\beta$, we have

$$e^\alpha{}_j(\mathfrak{p}) = \sum_{i=1}^{n} E^{\beta}{}_i{}^{\alpha}{}_j e^{\beta}{}_i(\mathfrak{p}),$$

where $E^{\beta}{}_i{}^{\alpha}{}_j$ is viewed as element of $\mathcal{O}_{\mathfrak{p}}$.

Let $E^{\beta\alpha}$ be the square matrix of order n whose entry in i-th row and j-th column is $E^{\beta}{}_i{}^{\alpha}{}_j$. Then we have the matrix product

$$E^{\alpha\gamma} E^{\gamma\beta} E^{\beta\alpha}$$

is equal to the unit matrix for any (α, β, γ), and that

$$E^{\alpha\beta}E^{\beta\alpha}$$

is also equal to the unit matrix. In particular, $\det E^{\beta\alpha}$ is a unit in the local ring of X at any point in $U_\alpha \cap U_\beta$. Reciprocally, if we are given a family of matrices with the above properties, then they serve as a system of transitions laws for a locally free sheaf defined on X which is unique up to isomorphism.

For each $\alpha \in J$, we choose a projective space P_α of dimension $n-1$ and fix a homogeneous coordinate system $\tau^\alpha_1, \cdots, \tau^\alpha_n$ for P_α. Form the product variety $U_\alpha \times P_\alpha$ and let π_α denote the coordinate projection from $U_\alpha \times P_\alpha$ onto U_α. Let $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ be the open set on $U_\alpha \times P_\alpha$ consisting of all points which π_α maps on $U_\alpha \cap U_\beta$. We are going to construct a bi-regular mapping $\phi_{\beta\alpha}$ of $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ onto $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$. Let $\mathfrak{e}$ be a point $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$, then $\phi_{\beta\alpha}(\mathfrak{e})$ is that point of $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$ such that

1) $\pi_\beta(\phi_{\beta\alpha}(\mathfrak{e})) = \pi_\alpha(\mathfrak{e}) = \mathfrak{p} \in U_\alpha \cap U_\beta$,

2) $\rho_1 \tau^\alpha_j(\mathfrak{e}) = \rho_2 \sum_{i=1}^{n} E^{\beta}{}_i{}^{\alpha}{}_j(\mathfrak{p}) \tau^\beta_i(\phi_{\beta\alpha}(\mathfrak{e}))$, $\qquad 1 \leqq j \leqq n$.

(ρ_1, ρ_2 are constants not both zero, and $E^{\beta}{}_i{}^{\alpha}{}_j(\mathfrak{p})$ is the value of the function $E^{\beta}{}_i{}^{\alpha}{}_j$ at $\mathfrak{p}$.) Since $\det E^{\beta\alpha}$ is a unit in the local ring $\mathfrak{O}_\mathfrak{p}$ of X at any point $\mathfrak{p}$ of $U_\alpha \cap U_\beta$, it follows that $\phi_{\beta\alpha}(\mathfrak{e})$ is a uniquely determined point on $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$.

From the properties of the family of matrices $\{E^{\beta\alpha}\}_{(\alpha,\beta)\in J\times J}$, we readily obtain the following:

(1) $\phi_{\beta\alpha}$ is a bi-regular mapping from $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ onto $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$;

(2) $\phi_{\alpha\beta}\phi_{\beta\alpha}$ is the identity map of $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$;

(3) $\phi_{\alpha\gamma}\phi_{\gamma\beta}\phi_{\beta\alpha}$ is the identity map of $\pi_\alpha^{-1}(U_\alpha \cap U_\beta \cap U_\gamma)$;

(4) The restriction of $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ and $\pi_\beta\phi_{\beta\alpha}$ are equal regular mappings of $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ onto $U_\alpha \cap U_\beta$. Now for each ordered pair (α, β), identify the open set $\pi_\alpha^{-1}(U_\alpha \cap U_\beta)$ on $U_\alpha \times U_\beta$ with the open set $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$ on $U_\beta \times P_\beta$ according to the rule that $\mathfrak{e}$ and $\phi_{\beta\alpha}(\mathfrak{e})$ are identical. The consistency of this method of identification follows from our previous assertions, and the set of points so obtained is covered by the family of subsets $\{U_\alpha \times P_\alpha\}_{\alpha\in J}$. We equip this set with the structure of an algebraic variety $\mathfrak{B}(E)$ such that

1) $\{U_\alpha \times P_\alpha\}$ forms a covering of $\mathfrak{B}(E)$ by open sets,

2) The algebro-geometric structure of $U_\alpha \times P_\alpha$ as product variety coincides with the one it inherits as open set on $\mathfrak{B}(E)$. We define a regular rational mapping π_E from $\mathfrak{B}(E)$ onto X according to the rule that the restriction of π_E to $U_\alpha \times P_\alpha$ is equal to π_α. π_E equips $\mathfrak{B}(E)$ with the structure of a projective fibre bundle over X with a projective space of dimension $n-1$ for fiber. Eventually, we shall prove that $\mathfrak{B}(E)$ admits a bi-regular mapping onto a projective model.

$\mathfrak{B}(E)$ is called the "dual projective bundle" of the sheaf E; it is independent of the particular family of generating sections utilized in its construction. Indeed, let $e'_1, \cdots, e'_n$ be sections of E over an open set U' on X such that $e'_1(\mathfrak{p}), \cdots, e'_n(\mathfrak{p})$ freely generate $E_\mathfrak{p}$ at each point $\mathfrak{p}$ on U'. Let P' be a projective space of dimension $n-1$ and fix a homogeneous coordinate system $\tau'_1, \cdots, \tau'_n$ for P'. Form the product variety $U' \times P'$ and let π' be the coordinate projection from $U' \times P'$ onto U'. We shall construct a bi-regular mapping ϕ' of $U' \times P'$ onto the open set $\pi_E^{-1}(U')$. Now for any $\alpha \in J$,

$$e'_j = \sum_{i=1}^{n} E^\alpha{}_i{}'_j e^\alpha{}_i, \qquad 1 \leqq j \leqq n,$$

where $E^\alpha{}_i{}'_j$ is regular on $U' \cap U_\alpha$ for all $1 \leqq i, j \leqq n$. Furthermore, $\det E^{\alpha\prime}$ is a unit in the local ring of X at any point $\mathfrak{p}$ on $U' \cap U_\alpha$. If $\mathfrak{e}'$ is a point on $(\pi')^{-1}(U' \cap U_\alpha)$, then $\phi'(\mathfrak{e}')$ is that point of $\mathfrak{B}(E)$ such that

1) $\pi_E(\phi'(\mathfrak{e}')) = \pi'(\mathfrak{e}') = \mathfrak{p} \in U' \cap U_\alpha$,

2) $\rho_1 \tau'_j(\mathfrak{e}') = \rho_2 \sum_{i=1}^{n} E^\alpha{}_i{}'_j(\mathfrak{p}) \tau^\alpha{}_i(\phi'(\mathfrak{e}')), \qquad 1 \leqq j \leqq n.$

One easily checks that $\phi'(\mathfrak{e}')$ does not depend upon the particular α such that $\mathfrak{e}' \in (\pi')^{-1}(U' \cap U_\alpha)$. We shall say that the sections $(e^\alpha{}_1, \cdots, e^\alpha{}_n)$ are paired with the homogeneous coordinates $\tau^\alpha{}_1, \cdots, \tau^\alpha{}_n$ with reference to the construction of $\mathfrak{B}(E)$. The above argument shows that for any generating sections $e'_1, \cdots, e'_n$ of E over U', there are uniquely determined homogeneous coordinates on $\pi_E^{-1}(U')$ which are paired with $e'_1, \cdots, e'_n$ with reference to the construction of $\mathfrak{B}(E)$. Furthermore, any different construction of the "dual projective bundle" is in bi-regular correspondence with $\mathfrak{B}(E)$ in exactly one way which conserves the pairing of sections and homogeneous coordinates.

§4. For any $\alpha \in J$, $1 \leqq i \leqq n$, let $U_{\alpha,i}$ be the open set on $\mathfrak{B}(E)$ consisting of all points $\mathfrak{e}$ on $U_\alpha \times P_\alpha$ $(= \pi_E^{-1}(U_\alpha))$ such that $\tau^\alpha{}_i(\mathfrak{e}) \neq 0$. The

family $\{U_{\alpha,i}\}_{\alpha\in J,1\leqq i\leqq n}$ forms an open covering of $\mathfrak{B}(E)$. For any ordered pair of couples (α,i), (β,j), we set

$$f^{\beta}{}_{j}{}^{\alpha}{}_{i}=\sum_{h=1}^{n}E^{\beta}{}_{h}{}^{\alpha}{}_{i}\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j}.$$

Observe that $f^{\beta}{}_{j}{}^{\alpha}{}_{i}$ is regular on $U_{\alpha,i}\cap U_{\beta,j}$. We also have that

$$f^{\beta}{}_{j}{}^{\alpha}{}_{i}\tau^{\alpha}{}_{k}/\tau^{\alpha}{}_{i}=\sum_{h=1}^{n}E^{\beta}{}_{h}{}^{\alpha}{}_{k}\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j},\qquad 1\leqq k\leqq n, \tag{1}$$

which proves that $f^{\beta}{}_{j}{}^{\alpha}{}_{i}$ vanishes at no point of $U_{\alpha,i}\cap U_{\beta,j}$, and that

$$f^{\alpha}{}_{i}{}^{\gamma}{}_{k}f^{\gamma}{}_{k}{}^{\beta}{}_{j}f^{\beta}{}_{j}{}^{\alpha}{}_{i}=1,$$
$$f^{\alpha}{}_{i}{}^{\beta}{}_{j}f^{\beta}{}_{j}{}^{\alpha}{}_{i}=1.$$

We construct a locally free sheaf $B(E)$ of dimension one defined on $\mathfrak{B}(E)$ as follows. The restriction of $B(E)$ to any $U_{\alpha,i}$ is a free sheaf of dimension one generated by a section $E[\alpha,i]$. On $U_{\alpha,i}\cap U_{\beta,j}$, we have the transition law

$$E[\alpha,i]=f^{\beta}{}_{j}{}^{\alpha}{}_{i}E[\beta,j].$$

Consider the sheaf $\pi_E E$, the reciprocal image of E by π_E, and for notational convenience, set

$${}^*E=\pi_E E.$$

The restriction of ${}^*E$ to any $U_\alpha\times P_\alpha$ is a free sheaf of dimension n generated by the sections $\pi_E{}^{-1}e^{\alpha}{}_{1},\cdots,\pi_E{}^{-1}e^{\alpha}{}_{n}$ which are the reciprocal images of $e^{\alpha}{}_{1},\cdots,e^{\alpha}{}_{n}$, and we set

$${}^*e^{\alpha}{}_{i}=\pi_E{}^{-1}e^{\alpha}{}_{i},\qquad 1\leqq i\leqq n.$$

For each (α,i), let $S_{\alpha,i}$ be the subsheaf of the restriction of ${}^*E$ to $U_{\alpha,i}$ which is generated by the sections

$${}^*e^{\alpha}{}_{h}-\tau^{\alpha}{}_{h}/\tau^{\alpha}{}_{i}{}^*e^{\alpha}{}_{i},\qquad 1\leqq h\leqq n.$$

$S_{\alpha,i}$ is a free sheaf of dimension $n-1$ defined on $U_{\alpha,i}$. Now

$${}^*e^{\alpha}{}_{h}=\sum_{j=1}^{n}E^{\beta}{}_{j}{}^{\alpha}{}_{h}{}^*e^{\beta}{}_{j},\qquad 1\leqq h\leqq n,$$

on $\pi_E{}^{-1}(U_\alpha\cap U_\beta)$, which leads to

$${}^*e^{\alpha}{}_{h}-\tau^{\alpha}{}_{h}/\tau^{\alpha}{}_{i}{}^*e^{\alpha}{}_{i}=\sum_{k=1}^{n}(E^{\beta}{}_{k}{}^{\alpha}{}_{h}-E^{\beta}{}_{k}{}^{\alpha}{}_{i}\tau^{\alpha}{}_{h}/\tau^{\alpha}{}_{i}){}^*e^{\beta}{}_{k},$$

and making use of (1), we obtain

$$ {}^*e^{\alpha}{}_{h} - \tau^{\alpha}{}_{h}/\tau^{\alpha}{}_{i}\,{}^*e^{\alpha}{}_{i} = (f^{\beta}{}_{j}{}^{\alpha}{}_{i})^{-1} \sum_{k=1}^{n} \mathcal{E}^{\beta}{}_{ji;}{}^{\alpha}{}_{kh}\,({}^*e^{\beta}{}_{k} - \tau^{\beta}{}_{k}/\tau^{\beta}{}_{j}\,{}^*e^{\beta}{}_{j}), $$

where

$$ (2) \quad \mathcal{E}^{\beta}{}_{ji;}{}^{\alpha}{}_{kh} = E^{\beta}{}_{k}{}^{\alpha}{}_{h}\Big(\sum_{l=1}^{n} E^{\beta}{}_{l}{}^{\alpha}{}_{i}\tau^{\beta}{}_{l}/\tau^{\beta}{}_{j}\Big) - E^{\beta}{}_{k}{}^{\alpha}{}_{i}\Big(\sum_{l=1}^{n} E^{\beta}{}_{l}{}^{\alpha}{}_{h}\tau^{\beta}{}_{l}/\tau^{\beta}{}_{j}\Big) = E^{\beta}{}_{k}{}^{\alpha}{}_{h}f^{\beta}{}_{j}{}^{\alpha}{}_{i} - E^{\beta}{}_{k}{}^{\alpha}{}_{i}f^{\beta}{}_{j}{}^{\alpha}{}_{h}. $$

We have proved that the restriction of $S_{\alpha,i}$ to $U_{\alpha,i} \cap U_{\beta,j}$ is equal to the restriction of $S_{\beta,j}$ to $U_{\alpha,i} \cap U_{\beta,j}$. Thus we obtain a locally free sheaf δE of dimension $n-1$ defined on $\mathcal{B}(E)$ whose restriction to any $U_{\alpha,i}$ is the sheaf $S_{\alpha,i}$. δE is called the "derived sheaf" of the sheaf E. The sheaf δE is intrinsically associated with E since its restriction $S_{\alpha,i}$ to any $U_{\alpha,i}$ depends upon the pairing of $e^{\alpha}{}_{1}, \cdots, e^{\alpha}{}_{n}$ with $\tau^{\alpha}{}_{1}, \cdots, \tau^{\alpha}{}_{n}$, which is an intrinsic property of $\mathcal{B}(E)$.

The restriction of ${}^*E$ to any $U_{\alpha,i}$ is a free sheaf of dimension n generated by the sections

$$ {}^*e^{\alpha}{}_{i}, $$

$$ {}^*e^{\alpha}{}_{h} - \tau^{\alpha}{}_{h}/\tau^{\alpha}{}_{i}\,{}^*e^{\alpha}{}_{i}, \qquad h \neq i,\ 1 \leqq h \leqq n. $$

We construct a homomorphism of ${}^*E$ onto $B(E)$ according to the rule that its restriction to any $U_{\alpha,i}$ is described by

$$ {}^*e^{\alpha}{}_{i} \rightarrow E[\alpha, i], $$

$$ {}^*e^{\alpha}{}_{h} - (\tau^{\alpha}{}_{h}/\tau_{\alpha i})\,{}^*e^{\alpha}{}_{i} \rightarrow 0. $$

To prove that the homomorphism is properly defined, we observe that

$$ {}^*e^{\alpha}{}_{i} = \sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}\,{}^*e^{\beta}{}_{h} = \Big(\sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j}\Big)\,{}^*e^{\beta}{}_{j} + \sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}\,({}^*e^{\beta}{}_{h} - \tau^{\beta}{}_{h}/\tau^{\beta}{}_{j}\,{}^*e^{\beta}{}_{j}) $$

on $U_{\alpha,i} \cap U_{\beta,j}$, and that

$$ E[\alpha, i] = \Big(\sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j}\Big)E[\beta, j]. $$

The kernel of this homomorphism is the sheaf δE; consequently, we can identify $B(E)$ with the quotient sheaf of ${}^*E$ modulo δE, and, as such, $B(E)$ intrinsically depends upon E and we call $B(E)$ the "basic sheaf" of E. The exact sequence

$$ (3) \qquad 0 \rightarrow \delta E \rightarrow \pi_{B}E \rightarrow B(E) \rightarrow 0 $$

of locally free sheaves defined on $\mathcal{B}(E)$ is called the "derived sequence" of the sheaf E.

We iterate this process and obtain the derived sheaf of δE which is a locally free sheaf of dimension $n-2$ defined on $\mathcal{B}(\delta E)$, and which we denote as $\delta_2 E$ and call it the second derived sheaf of E. Thus we have the exact sequence

$$0 \to \delta_2 E \to \pi_{\delta E}\delta E \to B(\delta E) \to 0$$

of locally free sheaves defined on $\mathcal{B}(\delta E)$. Repeating the process, we obtain for $1 \leqq s \leqq n-1$ the exact sequence

$$(3') \qquad 0 \to \delta_s E \to \pi_{\delta_{s-1}E}\delta_{s-1}E \to B(\delta_{s-1}E) \to 0$$

of locally free sheaves defined on $\mathcal{B}(\delta_{s-1}E)$. The s-th derived sheaf $\delta_s E$ is a locally free sheaf of dimension $n-s$ defined on $\mathcal{B}(\delta_{s-1}E)$.

Let ${}^{\#}E$ (resp. ${}^{\#}B(E)$) denote the reciprocal image of E (resp. $B(E)$) with respect to the regular map from $\mathcal{B}(\delta_{n-2}E)$ onto X (resp. $\mathcal{B}(E)$) which is the composite map $\pi_E \circ \cdots \circ \pi_{\delta_{n-2}E}$ (resp. $\pi_{\delta_1 E} \circ \cdots \circ \pi_{\delta_{n-2}E}$). Similarly, let ${}^{\#}\delta_s E$ (resp. ${}^{\#}B(\delta_s E)$) denote the reciprocal image of $\delta_s E$ (resp. $B(\delta_s E)$) with respect to the regular map from $\mathcal{B}(\delta_{n-2}E)$ onto $\mathcal{B}(\delta_{s-1}E)$ (resp. $\mathcal{B}(\delta_s E)$). ${}^{\#}E$ is a locally free sheaf of dimension n defined on $\mathcal{B}(\delta_{n-2}E)$ and it has a composition series

$$(4) \qquad {}^{\#}E \supset {}^{\#}\delta_1 E \supset \cdots \supset {}^{\#}\delta_{n-1}E = \delta_{n-1}E.$$

${}^{\#}\delta_s E$ is a locally free sheaf of dimension $n-s$ defined on $\mathcal{B}(\delta_{n-2}E)$ and the quotient sheaf ${}^{\#}\delta_s E/{}^{\#}\delta_{s-1}E$ is canonically isomorphic with ${}^{\#}B(\delta_s E)$.

§ 5. For $2 \leqq s \leqq n$, we form the sheaf $\wedge^s E$ which is the exterior product of E taken s times, and we set $\wedge^1 E = E$. $\wedge^s E$ is a locally free sheaf of dimension $n!/s!(n-s)!$ defined on X. The stalk $\wedge^s E_{\mathfrak{p}}$ at any point $\mathfrak{p} \in U_\alpha$ is the exterior product of $E_{\mathfrak{p}}$ taken s times and its is freely generated by

$$e^\alpha{}_{h_1}(\mathfrak{p}) \wedge \cdots \wedge e^\alpha{}_{h_s}(\mathfrak{p}), \qquad 1 \leqq h_1 < \cdots < h_s \leqq n.$$

We set

$$e^\alpha{}_{h_1\cdots h_s}(\mathfrak{p}) = e^\alpha{}_{h_1}(\mathfrak{p}) \wedge \cdots \wedge e^\alpha{}_{h_s}(\mathfrak{p})$$

and agree that $e^\alpha{}_{h_1\cdots h_s}(\mathfrak{p})$ is defined for all sequences of s integers from $\{1,\cdots,n\}$ but that it is strictly skew-symmetric in its subscripts (hence zero if two subscripts are equal). The restriction of $\wedge^s E$ to any U_α is a free sheaf generated (freely) by sections

$$e^\alpha{}_{h_1\cdots h_s}, \qquad 1 \leqq h_1 < \cdots < h_s \leqq n,$$

where the image of $e^\alpha{}_{h_1\cdots h_s}$ in $(\wedge^s E)_{\mathfrak{p}}$ is $e^\alpha{}_{h_1\cdots h_s}(\mathfrak{p})$.

Consider the reciprocal image $\pi_E \wedge^s E$ of $\wedge^s E$ with respect to π_E and set

$$*E^s = \pi_E \wedge^s E.$$

The restriction of $*E^s$ to any $U_\alpha \times P_\alpha$ is a free sheaf of dimension $n!/s!(n-s)!$ generated by the sections

$$*e^\alpha{}_{h_1\cdots h_s}, \qquad 1 \leqq h_1 < \cdots < h_s \leqq n,$$

where $*e^\alpha{}_{h_1\cdots h_s}$ is the reciprocal image of $e^\alpha{}_{h_1\cdots h_s}$. From our canonical isomorphism of δE into $*E$, we construct a canonical isomorphism of $\wedge^s(\delta E)$ into $\wedge^s(*E) = *E^s = \pi_E \wedge^s E$. For the restriction of δE to any U_{α,h_0} is a free sheaf generated by the sections

$$*e^\alpha{}_h - (\tau^\alpha{}_h/\tau^\alpha{}_{h_0}) *e^\alpha{}_{h_0}, \qquad h \neq h_0, 1 \leqq h \leqq n;$$

hence the restriction of $\wedge^s(\delta E)$ to U_{α,h_0} is a free sheaf of dimension $(n-1)!/s!(n-1-s)!$ generated by the sections

$$(*e^\alpha{}_{h_1} - (\tau^\alpha{}_{h_1}/\tau^\alpha{}_{h_0}) *e^\alpha{}_{h_0}) \wedge \cdots \wedge (*e^\alpha{}_{h_s} - (\tau^\alpha{}_{h_s}/\tau^\alpha{}_{h_0}) *e^\alpha{}_{h_0})$$

for all $1 \leqq h_1 < \cdots < h_s \leqq n$ for all $1 \leqq i \leqq s$. The isomorphism from $\wedge^s(\delta E)$ into $\wedge^s(*E)$ is defined according to the rule that its restriction to U_{α,h_0} is given by

$$(*e^\alpha{}_{h_1} - \tau^\alpha{}_{h_1}/\tau^\alpha{}_{h_0} *e^\alpha{}_{h_0}) \wedge \cdots \wedge (*e^\alpha{}_{h_s} - \tau^\alpha{}_{h_s}/\tau^\alpha{}_{h_0} *e^\alpha{}_{h_0}) \rightarrow$$

$$*e^\alpha{}_{h_1\cdots h_s} + \sum_{p=1}^{s} (-1)^p (\tau^\alpha{}_{h_p}/\tau^\alpha{}_{h_0}) *e^\alpha{}_{h_0 h_1 \cdots \hat{h}_p \cdots h_s}.$$

($*e^\alpha{}_{h_0 h_1 \cdots \hat{h}_p \cdots h_s}$ is obtained by suppressing the index h_p from the sequence of $s+1$ interegers $h_0, \cdots, h_s$.) Indeed, the restriction of $\wedge^s(*E)$ to U_{α,h_0} is freely generated by the sections

$$*e^\alpha{}_{h_0 k_1 \cdots k_{s-1}}, \qquad 1 \leqq k_1 < \cdots < k_{s-1} \leqq n, k_i \neq h_0,$$

$$*e^\alpha{}_{h_1\cdots h_s} + \sum_{p=1}^{s} (-1)^p (\tau^\alpha{}_{h_p}/\tau^\alpha{}_{h_0}) *e^\alpha{}_{h_0 h_1 \cdots \hat{h}_p \cdots h_s},$$

$$1 \leqq h_1 < \cdots < h_s \leqq n, h_i \neq h_0,$$

and we can view $\wedge^s(\delta E)$ as a subsheaf $\wedge^s(*E)$.

It is evident that the quotient sheaf of $\wedge^s(*E)$ modulo $\wedge^s(\delta E)$ is a locally free sheaf defined on $\mathfrak{B}(E)$. We shall construct a canonical isomorphism of this quotient sheaf into the tensor product sheaf $\wedge^{s-1}(*E) \otimes B(E)$. For the restriction of $\wedge^{s-1}(*E) \otimes B(E)$ to U_{α,h_0} is a free sheaf generated by

$$*e^\alpha{}_{h_0 k_1 \cdots k_{s-2}} \otimes E[\alpha, h_0], \qquad 1 \leqq k_1 < \cdots < k_{s-2} \leqq n, k_i \neq h_0,$$

$$*e_{k_1 \cdots k_{s-1}} \otimes E[\alpha, h_0] - \sum_{p=1}^{s-1} (-1)^p (\tau^\alpha{}_{h_p}/\tau^\alpha{}_{h_0}) *e_{h_0 k_1 \cdots \hat{k}_p \cdots k_{s-1}} \otimes E[\alpha, h_0]$$

for all $1 \leqq k_1 < \cdots < k_{s-1} \leqq n-1$, $k_i \neq h_0$. We construct a homomorphism from $\wedge^s(*E)$ into $\wedge^{s-1}(*E) \otimes B(E)$ according to the rule that its restriction to U_{α,h_0} is given by

$$e_{h_1 \cdots h_s} \to \sum_{p=1}^{s} (-1)^{p-1} (\tau^{\alpha}{}_{h_p}/\tau^{\alpha}{}_{h_0})^* e^{\alpha}{}_{h_1 \cdots \hat{h}_p \cdots h_s} \otimes E[\alpha, h\ ].$$

One checks that we have properly defined a homomorphism, that its kernel is $\wedge^s(\delta E)$, and that the image of $\wedge^s(*E)$ is $\wedge^{s-1}(\delta E) \otimes B(E)$. Thus we have the exact sequence

$$(5) \qquad 0 \to \wedge^s(\delta E) \to \wedge^s(*E) \to \wedge^{s-1}(\delta E) \otimes B(E) \to 0,$$

which specializes to the derived sequence

$$0 \to \delta E \to *E \to B(E) \to 0$$

for $s = 1$.

Let $B^{-1}(E)$ be the reciprocal sheaf of $B(E)$. $B^{-1}(E)$ is a locally free sheaf of dimension one defined on $\mathfrak{B}(E)$. The restriction of $B^{-1}(E)$ to any $U_{\alpha,i}$ is a free sheaf of dimension one generated by a section $E^{-1}[\alpha, i]$ over $U_{\alpha,i}$, and we have the transition law

$$E^{-1}[\alpha, i] = (f^{\beta}{}_j{}^{\alpha}{}_i)^{-1} E^{-1}[\beta, j]$$

on $U_{\alpha,i} \cap U_{\beta,j}$, where $f^{\beta}{}_j{}^{\alpha}{}_i$ is, as before, equal to

$$\sum_{h=1}^{n} E^{\beta}{}_h{}^{\alpha}{}_i \tau^{\beta}{}_h / \tau^{\beta}{}_j.$$

The tensor product

$$B^{-1}(E) \otimes B(E)$$

is canonically isomorphic to the sheaf $\mathcal{O}_{\mathfrak{B}(E)}$ of local rings on $\mathfrak{B}(E)$, since we have

$$E[\alpha, i] = f^{\beta}{}_j{}^{\alpha}{}_i E[\beta, j]$$

on any $U_{\alpha,i} \cap U_{\beta,j}$. Taking tensor products with the sheaf $B^{-1}(E)$ a suitable number of times, the previous exact sequences (5) can be merged into a single exact sequence

$$(6) \qquad 0 \to \wedge^n(*E \otimes B^{-1}(E)) \to \cdots \to \wedge^s(*E \otimes B^{-1}(E)) \to \cdots \to \mathcal{O}_{\mathfrak{B}(E)} \to 0$$

of locally free sheaves defined on $\mathfrak{B}(E)$. The kernel of

$$\wedge^s(*E \otimes B^{-1}(E)) \to \wedge^{s-1}(*E \otimes B^{-1}(E))$$

is the sheaf

$$\wedge^s(\delta E \otimes B^{-1}(E)).$$

Set

$$B^{-k}(E) = \underbrace{B^{-1}(E) \otimes \cdots \otimes B^{-1}(E)}_{k\text{-times}}.$$

Then we have the exact sequence

$$(7) \quad 0 \to \wedge^n(*E) \otimes B^{-(n-s)}(E) \to \cdots \to \wedge^{s+k}(*E) \otimes B^{-k}(E) \to \cdots$$
$$\cdots \to \wedge^{s+1}(*E) \otimes B^{-1}(E) \to \wedge^s(\delta E) \to 0.$$

Let $\Omega^1{}_X$ denote the sheaf of germs of regular differentials of degree one on X, and similarly we have $\Omega^1{}_{\mathfrak{B}(E)}$. Set $*\Omega^1{}_X$ equal to the sheaf $\pi_E\Omega^1{}_X$, the reciprocal image of $\Omega^1{}_X$ with respect to π_E. Then there is a canonical isomorphism of $*\Omega^1{}_X$ into $\Omega^1{}_{\mathfrak{B}(E)}$, and the quotient sheaf is easily seen to be isomorphic to the sheaf $\delta E \otimes B^{-1}(E)$. Thus we have the exact sequence

$$(8) \qquad 0 \to \pi_E\Omega^1{}_X \to \Omega^1{}_{\mathfrak{B}(E)} \to \delta E \otimes B^{-1}(E) \to 0.$$

§ 6. Let z be a rational function on $\mathfrak{B}(E)$ with the property that z is regular on some $U_\alpha \times P_\alpha$. We shall prove that z is the reciprocal image of a regular function on U_α. For z is expressible as a quotient P/Q where P, Q are homogeneous polynomials, of equal degree of homogeneity say s, in $\tau^\alpha{}_1, \cdots, \tau^\alpha{}_n$ with coefficients in the field of rational functions on X. We can suppose that P, Q are relatively prime polynomials in $\tau^\alpha{}_1, \cdots, \tau^\alpha{}_n$, and, since U_α is an affine model, we can suppose that the coefficients of P and Q are the reciprocal images of regular functions on U_α. If $s > 0$, then we can find a point $\mathfrak{e}$ on $U_\alpha \times P_\alpha$ such that $P(\mathfrak{e}) \neq 0$ and $Q(\mathfrak{e}) = 0$ since P and Q are relatively prime. Consequently, our assumption that z is regular on $U_\alpha \times P_\alpha$ forces $s = 0$ so that z is then the reciprocal image of a regular function on U_α.

It now follows that the canonical homomorphism from $\Gamma(E, U_\alpha)$ into $\Gamma(\pi_E E, U_\alpha \times P_\alpha)$ is an isomorphism of the former onto the latter. The restriction to $U_{\alpha,i}$ of the homomorphism from $\Gamma(\pi_E E, U_\alpha \times P_\alpha)$ into $\Gamma(B(E), U_\alpha \times P_\alpha)$ is described by

$$\pi_E{}^{-1}e^\alpha{}_h \to (\tau^\alpha{}_h/\tau^\alpha{}_i)E[\alpha, i], \qquad 1 \leqq h \leqq n.$$

This induces a homomorphism from $\Gamma(E, U_\alpha)$ into $\Gamma(B(E), U_\alpha \times P_\alpha)$ which sends the section

$$c_1 e^\alpha{}_1 + \cdots + c_n e^\alpha{}_n$$

($c_1, \cdots, c_n$ regular on U_α) into the section of $B(E)$ over $U_\alpha \times P_\alpha$ whose restriction to any $U_{\alpha,i}$ is the section

$$\left(\sum_{h=1}^{n} c_h \tau^\alpha{}_h/\tau^\alpha{}_i\right)E[\alpha, i].$$

This homomorphism is an isomorphism from $\Gamma(E, U_\alpha)$ into $\Gamma(B(E), U_\alpha \times P_\alpha)$. For $B(E)$ is a locally free sheaf, and if the image section is zero, then

$$\sum_{h=1}^{n} c_h \tau^\alpha{}_h/\tau^\alpha{}_i$$

is the rational function zero. But $\tau^\alpha{}_h/\tau^\alpha{}_i$, $1 \leqq h \leqq n$, $h \neq i$, are $n-1$ algebraically independent functions over the field of rational functions on X which forces $c_1, \cdots, c_n$ equal to zero.

We shall strengthen the last remark by proving that $\Gamma(E, U_\alpha)$ is mapped isomorphically onto $\Gamma(B(E), U_\alpha \times P_\alpha)$. Let σ be a section of $\Gamma(B(E), U_\alpha \times P_\alpha)$. The restriction of σ to any $U_{\alpha,i}$ is equal to

$$z_i E[\alpha, i],$$

where z_i is a regular function on $U_{\alpha,i}$. On $U_{\alpha,i} \cap U_{\alpha,j}$, we have

$$z_i(\tau^\alpha{}_i/\tau^\alpha{}_j) = z_j$$

since

$$E[\alpha, i] = (\tau^\alpha{}_i/\tau^\alpha{}_j) E[\alpha, j],$$

and z_j is regular on $U_{\alpha,j}$. For i fixed, we have

$$z_i = P/Q,$$

where P, Q are homogeneous polynomials, of equal degree of homogeneity s, in $\tau^\alpha{}_1, \cdots, \tau^\alpha{}_n$; the coefficients are regular functions on U_α and P, Q are relatively prime as polynomials over the field of rational functions on X. Now z_i is regular on $U_{\alpha,i}$ which forces Q to be a monomial

$$A(\tau^\alpha{}_i)^s,$$

where A is regular on U. Otherwise, since P and Q are relatively prime, there would exist a point e on $U_{\alpha,i}$ such that $Q(\mathrm{e}) = 0$ and $P(\mathrm{e}) \neq 0$, which contradicts the assumption that z_i is regular. Choose $j \neq i$ (there is nothing to prove if $n = 1$) and observe that

$$z_j = (\tau^\alpha{}_i/\tau^\alpha{}_j) P/Q = P/A\tau^\alpha{}_j(\tau^\alpha{}_i)^{s-1}$$

is regular on $U_{\alpha,j}$. Since $\tau^\alpha{}_i$ does not divide P, the only possibilities are $s = 0$ or $s = 1$. If $s = 0$, then z_i is the reciprocal image of a regular function z_i on U_α and σ is the image of the section $z_i e^\alpha{}_i$. If $s = 1$, then

$$z_i = \sum_{h=1}^{n} c_h(\tau^\alpha{}_h/\tau^\alpha{}_i),$$

where $c_1, \cdots, c_n$ are regular on U_α and σ is the image of the section

$$\sum_{h=1}^{n} c_h e^\alpha{}_h.$$

Since $E, \pi_E E$, and $B(E)$ are locally free sheaves it follows that $\Gamma(E,X)$, $\Gamma(\pi_E E, \mathfrak{B}(E))$ and $\Gamma(B(E), \mathfrak{B}(E))$ are isomorphic. J. H. Sampson and the author have generalized this result by proving that cohomology modules $H^q(X,E)$, $H^q(\mathfrak{B}(E), \pi_E E)$, and $H^q(\mathfrak{B}(E), B(E))$ are isomorphic for all q.

§ 7. We shall borrow Andre Weil's construction of the basic characteristic class of a locally free sheaf of dimension one. We choose a family $\{g^\alpha{}_i\}$ of rational functions on $\mathfrak{B}(E)$, a function $g^\alpha{}_i$ for each (α, i), such that

1) $g^\alpha{}_i$ is not the function zero for each (α, i),

2) $f^{\beta}{}_j{}^\alpha{}_i g^\alpha{}_i = g^\beta{}_j$ for each (α, i), (β, j), where

$$f^{\beta}{}_j{}^\alpha{}_i = \sum_{h=1}^{n} E^{\beta}{}_h{}^\alpha{}_i (\tau^\beta{}_h / \tau^\beta{}_j).$$

Let W be a proper subvariety on $\mathfrak{B}(E)$ of highest possible dimension (i.e., if $\dim. X = r$, then $\dim. \mathfrak{B}(E) = r + n - 1$ and $\dim. W = r + n - 2$). Choose (α, i) such that the frontier of $U_{\alpha,i}$ does not contain W and define

$$\mathrm{ord}_W \{g\}$$

to be $\mathrm{ord}_W\, g^\alpha{}_i$, the order of the function $g^\alpha{}_i$ along W; there is no difficulty since $\mathfrak{B}(E)$ is a non-singular variety. Since $f^{\beta}{}_j{}^\alpha{}_i$ is regular and vanishes at no point of $U_{\alpha,i} \cap U_{\beta,j}$, it follows that

$$\mathrm{ord}_W\, g^\beta{}_j = \mathrm{ord}_W\, g^\alpha{}_i$$

if the frontier of $U_{\beta,j}$ does not contain W, so that $\mathrm{ord}_W \{g\}$ is well defined. Let $\vartheta_{\{g\}}$ be the divisor on $\mathfrak{B}(E)$ with

$$\vartheta_{\{g\}} = \sum{}_W (\mathrm{ord}_W \{g\})\, W,$$

where the sum is taken over all proper subvarieties W on $\mathfrak{B}(E)$ of highest possible dimension; there are, of course, only finitely many W such that $\mathrm{ord}_W \{g\} \neq 0$. Let $\{h^\alpha{}_i\}$ be some other family with the properties 1) and 2) above. Then there is a unique rational function f on $\mathfrak{B}(E)$, not the function zero, such that

$$f = g^\alpha{}_i / h^\alpha{}_i$$

for each (α, i), since $g^\alpha{}_i / h^\alpha{}_i = g^\alpha{}_j / h^\alpha{}_j$. The divisors $\vartheta_{\{g\}}$ and $\vartheta_{\{h\}}$ are linearly equivalent divisors on $\mathfrak{B}(E)$ and let $\Theta(E)$ denote the common divisor class of these divisors. $\Theta(E)$ is called the "basic divisor class" of the sheaf E. The basic sheaf $B(E)$ is isomorphic with the sheaf $\mathfrak{L}(-\vartheta)$ of germs of rational functions on $\mathfrak{B}(E)$ which are multiples of the divisor $-\vartheta$ where ϑ is any divisor of the class $\Theta(E)$. Consequently, we have that

$$B(E) = \mathcal{O}_{\mathfrak{B}(E)}(\Theta(E)).$$

The divisor $\vartheta_{\{g\}}$ is non-negative if and only if $g^{\alpha}{}_{i}$ is a regular function on $U_{\alpha,i}$ for every (α, i). In this situation, we obtain a non-zero element of the module $\Gamma(B(E), \mathfrak{B}(E))$. It is the global section of $B(E)$ whose restriction to any $U_{\alpha,i}$ is $g^{\alpha}{}_{i}E[\alpha, i]$. Reciprocally, to each non-zero global section of $B(E)$, there corresponds a unique non-negative divisor of the class $\Theta(E)$, and two non-zero sections of $B(E)$ correspond to the same divisor if and only if they are linearly dependent elements of $\Gamma(B(E), \mathfrak{B}(E))$ as vector space over the constant field. In view of the previous isomorphism of $\Gamma(E, X)$ onto $\Gamma(B(E), \mathfrak{B}(E))$, it follows that to each non-zero element of $\Gamma(E, X)$, there corresponds a unique non-negative divisor of $\Theta(E)$. If $\dim.\Gamma(E, X) > 0$, then $\Theta(E)$ contains positive divisors except for the case where E is equal to $\mathcal{O}_X$, for here $\Theta(\mathcal{O}_X)$ is the divisor class zero.

§ 8. Let D be a locally free sheaf of dimension one defined on X. It follows from our constructions that

$$\mathfrak{B}(D) = X,$$

and that

$$B(D) = D.$$

We can suppose that the previous covering $\{U_\alpha\}$ has the property that the restriction of D to any U_α is a free sheaf generated by a section d^α. On $U_\alpha \cap U_\beta$ we have the transition law

$$d^\alpha = D^{\beta\alpha} d^\beta,$$

where $D^{\beta\alpha}$ is regular and vanishes at no point of $U_\alpha \cap U_\beta$. Form the product cheaf $E \otimes D$. The restriction of $E \otimes D$ to any U_α is a free sheaf of dimension n generated by the sections

$$e^{\alpha}{}_{1} \otimes d^\alpha, \cdots, e^{\alpha}{}_{n} \otimes d.$$

On $U_\alpha \cap U_\beta$, wehave the transition laws

$$e^{\alpha}{}_{j} \otimes d^\alpha = D^{\beta\alpha} \sum_{i=1}^{n} E^{\beta}{}_{i}{}^{\alpha}{}_{j} e^{\beta}{}_{i} \otimes d^\beta, \qquad 1 \leqq i \leqq n.$$

There is an evident bi-regular mapping of $\mathfrak{B}(E \otimes D)$ onto $\mathfrak{B}(E)$ with the property that $\tau^{\alpha}{}_{1}, \cdots, \tau^{\alpha}{}_{n}$ corresponds to the homogeneous coordinate system paired with $e^{\alpha}{}_{1} \otimes d^\alpha, \cdots, e^{\alpha}{}_{n} \otimes d^\alpha$ for every $\alpha \in J$. Thus

$$\mathfrak{B}(E \otimes D) = \mathfrak{B}(E)$$

and

$$B(E \otimes D) = B(E) \otimes \pi_E D.$$

We also have that

$$\Theta(E \otimes D) = \Theta(E) + \pi_E{}^{*}\Theta(D),$$

where $\pi_E{}^*\Theta(D)$ is the reciprocal image of the divisor class $\Theta(D)$ with respect to π_E.

Let Y be a non-singular projective model, and let Φ be a regular mapping from Y into X. $\mathfrak{B}(\Phi E)$, the dual projective bundle of the reciprocal image sheaf ΦE, is identical to the induced bundle of $\mathfrak{B}(E)$ by the mapping Φ. Furthermore, $B(\Phi E)$ is the reciprocal image of $B(E)$ by the fiber mapping of $\mathfrak{B}(\Phi E)$ into $\mathfrak{B}(E)$ which covers the mapping Φ of Y into X, and $\Theta(\Phi E)$ is the divisor class on $\mathfrak{B}(\Phi E)$ which is the reciprocal image of $\Theta(E)$ by that fiber mapping.

§ 9. Let V be a non-singular subvariety of dimension d on the non-singular projective model X of dimension r. Consider the sheaf $\mathfrak{J}(X;V)$ of ideals determined by V on X; the stalk of $\mathfrak{J}(X;V)$ at any point $\mathfrak{p}$ on X is the ideal determined by V in the local ring $\mathfrak{O}_{\mathfrak{p}}$ of X at $\mathfrak{p}$. The induced sheaf of $\mathfrak{J}(X;V)$ on the subvariety V (i.e., the reciprocal image with respect to the identity map of V into X) is a locally free sheaf $N(X;V)$ of dimension $r-d$ defined on V which we call the sheaf of covariant normal vectors to V on X.

To prove the above assertions, we choose a covering $\mathfrak{U}=\{U_\alpha\}_{\alpha\in J}$ of X consisting of finitely many non-empty open set U_α such that each U_α admits a biregular correspondence onto an affine model and with the following additional properties:

1) If $U_\alpha \cap V$ is non-empty, then there exist $r-d$ regular functions $x^\alpha{}_1,\cdots,x^\alpha{}_{r-d}$ on U_α which generate the ideal determined by V in the local ring of X at any point on U_α;

2) For each point $\mathfrak{p}$ on U_α, the $r-d$ functions $x^\alpha{}_h - x^\alpha{}_h(\mathfrak{p})$, $1\leqq h\leqq r-d$, ($x^\alpha{}_h(\mathfrak{p})$ is the value of $x_{\alpha h}$ at $\mathfrak{p}$) can be extended to a basis of $r-d$ elements for the maximal prime ideal in the local ring of X at $\mathfrak{p}$;

3) Each $U_\alpha\cap U_\beta$ admits a bi-regular mapping onto an affine model. The possibility of choosing such a covering follows from the assumption that X and V are non-singular together with the quasi-compactness of the Zariski topology.

Let J^* be the subset of J consisting of those α such that $U_\alpha\cap V$ is non-empty. Given $\alpha,\beta\in J^*$, it is possible to choose regular functions $N^\beta{}_g{}^\alpha{}_h$, $1\leqq g,h\leqq r-d$, on $U_\alpha\cap U_\beta$ such that

$$x^\alpha{}_h=\sum_{g=1}^{r-d} N^\beta{}_g{}^\alpha{}_h x^\beta{}_g, \qquad 1\leqq h\leqq r-d,$$

for the restrictions of $x^\alpha{}_1, \cdots, x^\alpha{}_{r-d}$ to $U_\alpha \cap U_\beta$ generate the module of sections of $\mathfrak{J}(X;V)$ over $U_\alpha \cap U_\beta$ since that open set is an affine model. If $r-d>1$, then the $N^\beta{}_g{}^\alpha{}_h$ are not uniquely determined by the choice of the families $\{x^\alpha{}_1, \cdots, x^\alpha{}_{r-d}\}_{\alpha \in J^*}$; however, their induced functions on V, which we continue to denote as $N^\beta{}_g{}^\alpha{}_h$, are regular functions on $U_\alpha \cap U_\beta \cap V$ which are uniquely determined by the families $\{x^\alpha{}_1, \cdots, x^\alpha{}_{r-d}\}_{\alpha \in J^*}$.

Let $\mathfrak{p}$ be any point on V; say that $\mathfrak{p} \in U_\alpha \cap V$. The stalk $\mathfrak{J}(X;V)_\mathfrak{p}$ is the submodule of $\mathcal{O}_\mathfrak{p}$ generated by $x^\alpha{}_1, \cdots, x^\alpha{}_{r-d}$. Let $\mathcal{O}(V,\mathfrak{p})$ denote the local ring of V at $\mathfrak{p}$; it is the residue class ring of $\mathcal{O}_\mathfrak{p}$ modulo the ideal $\mathfrak{J}(X;V)_\mathfrak{p}$ so that $\mathcal{O}(V;\mathfrak{p})$ carries the structure of an $\mathcal{O}_\mathfrak{p}$-module. Form the tensor product $\mathcal{O}(V;\mathfrak{p}) \otimes_{\mathcal{O}_\mathfrak{p}} \mathfrak{J}(X;V)_\mathfrak{p}$; it is the stalk $N(X;V)_\mathfrak{p}$ with the structure of an $\mathcal{O}_\mathfrak{p}$-module. We have that

$$1 \otimes x^\alpha{}_1, \cdots, 1 \otimes x^\alpha{}_{r-d}, \qquad 1 \in \mathcal{O}(V;\mathfrak{p}),$$

generate $N(X;V)_\mathfrak{p}$ as $\mathcal{O}(V,\mathfrak{p})$-module. Set

$$z^\alpha{}_h = 1 \otimes x^\alpha{}_h, \qquad 1 \leqq h \leqq r-d;$$

then we shall prove that $N(X;V)_\mathfrak{p}$ is the free $\mathcal{O}(V;\mathfrak{p})$-module generated by $z^\alpha{}_1, \cdots, z^\alpha{}_{r-d}$. Let F be a free $\mathcal{O}_\mathfrak{p}$-module of dimension $r-d$ with generators $f_1, \cdots, f_{r-d}$, and define a homomorphism from F onto $\mathfrak{J}(X;V)_\mathfrak{p}$ according to

$$f_h \to x^\alpha{}_h, \qquad 1 \leqq h \leqq r-d.$$

The kernel of this homomorphism is the submodule R of F generated by

$$x^\alpha{}_{h_1} f_{h_2} - x^\alpha{}_{h_2} f_{h_1}, \qquad 1 \leqq h_1 < h_2 \leqq r-d,$$

since $\mathfrak{p}$ is a simple point on V. From the exact sequence

$$0 \to R \to F \to \mathfrak{J}(X,V)_\mathfrak{p} \to 0$$

of $\mathcal{O}_\mathfrak{p}$-modules, we pass to the exact sequence of $\mathcal{O}(V;\mathfrak{p})$-modules

$$\mathcal{O}(V;\mathfrak{p}) \otimes R \to \mathcal{O}(V;\mathfrak{p}) \otimes F \to N(X;V)_\mathfrak{p} \to 0$$

which is obtained by tensorizing with $\mathcal{O}(V;\mathfrak{p})$ viewed as $\mathcal{O}_\mathfrak{p}$-module, and then viewing each module as $\mathcal{O}(V;\mathfrak{p})$-module. We have that $\mathcal{O}(V;\mathfrak{p}) \otimes F$ is a free $\mathcal{O}(V;\mathfrak{p})$-module generated by

$$1 \otimes f_h, \qquad 1 \leqq h \leqq r-d,$$

and that the homomorphism

$$\mathcal{O}(V;\mathfrak{p}) \otimes R \to \mathcal{O}(V;\mathfrak{p}) \otimes F$$

maps the first module onto the zero module since it sends each element

$$1 \otimes (x^{\alpha}{}_{h_1} f_{h_2} - x^{\alpha}{}_{h_2} f_{h_1}), \qquad 1 \leqq h_1 < h_2 \leqq r - d,$$

of $\mathcal{O}(V;\mathfrak{p}) \otimes R$ onto the element

$$x^{\alpha}{}_{h_1} \otimes f_{h_2} - x^{\alpha}{}_{h_2} \otimes f_{h_1} = 0$$

of $\mathcal{O}(V;\mathfrak{p}) \otimes F$. Consequently, we have the exact sequence

$$0 \rightarrow \mathcal{O}(V;\mathfrak{p}) \otimes F \rightarrow N(X;V)_{\mathfrak{p}} \rightarrow 0,$$

with

$$1 \otimes f_h \rightarrow z^{\alpha}{}_h, \qquad 1 - h - r - d;$$

and this proves that $N(X;V)$ is a locally free sheaf of dimension $r-d$ defined on V.

Thus the restriction of $N(X;V)$ to any $U_\alpha \cap V$ is a free sheaf dimension $r-d$ generated by sections

$$z^{\alpha}{}_h = 1 \otimes x^{\alpha}{}_h, \qquad 1 \leqq h \leqq r - d,$$

(where 1 is the section 1 of $\mathcal{O}_V$ over $U_\alpha \cap V$), and we have the transition laws

$$z^{\alpha}{}_h = \sum_{g=1}^{r-d} N^{\beta}{}_g{}^{\alpha}{}_h z^{\beta}{}_g, \qquad 1 \leqq h \leqq r - d,$$

on any $U_\alpha \cap U_\beta \cap V$.

The sheaf $N(X;V)$ depends solely upon the embedding of V in X; it is essentially the sheaf of germs of cross sections of the vector bundle of differentials on X which are normal to the subvariety V (i.e., the dual space to the subspace of tangent vectors of V in the tangent space of X). We also have the exact sequence

$$0 \rightarrow N(X;V) \rightarrow (\Omega^1{}_X)' \rightarrow \Omega^1{}_V \rightarrow 0$$

of locally free sheaves defined on V. $(\Omega^1{}_X)'$ is the induced sheaf on V of the sheaf of differentials of degree one on X; and $\Omega^1{}_V$ is the sheaf of differentials of degree one on V.

§ 10. We shall review the classical construction of the variety X^* which is obtained from monoidal transformation of X centered on V; it is related to the construction of the sheaf $N(X;V)$ in § 9. Choose a projective space $P^*{}_\alpha$ of dimension $r-d-1$ for each $\alpha \in J^*$ and fix a homogeneous coordinate system $\zeta^{\alpha}{}_1, \cdots, \zeta^{\alpha}{}_{r-d}$ on $P^*{}_\alpha$. Form the product variety $U_\alpha \times P^*{}_\alpha$ and let $U^*{}_\alpha$ denote the set of all points $\mathfrak{p}^*$ on $U_\alpha \times P^*{}_\alpha$ such that

$$x^{\alpha}{}_{h_1}(\mathfrak{p}^*)\zeta^{\alpha}{}_{h_2}(\mathfrak{p}^*) - x^{\alpha}{}_{h_2}(\mathfrak{p}^*)\zeta^{\alpha}{}_{h_1}(\mathfrak{p}^*) = 0$$

for all $1 \leqq h_1, h_2 \leqq r-d$. U^*_α is a non-singular subvariety of dimension r on $U_\alpha \times P^*_\alpha$. Let $\mathfrak{p}^*$ be any point on U^*_α, and say that $\zeta^\alpha_{h_0}(\mathfrak{p}^*) \neq 0$ for some h_0, $1 \leqq h_0 \leqq r-d$. Then the $r-d-1$ functions.

$$x^\alpha_h - x^\alpha_{h_0}(\zeta^\alpha_h/\zeta^\alpha_{h_0}), \qquad 1 \leqq h \leqq r-d, h \neq h_0,$$

generate the ideal determined by U^*_α in the local ring of $U_\alpha \times P^*_\alpha$ at $\mathfrak{p}^*$; this proves that U^*_α is a non-singular subvariety of dimension r on $U_\alpha \times P^*_\alpha$.

Let Φ_α denote the restriction to U^*_α of the coordinate projection from $U_\alpha \times P^*_\alpha$ onto U_α. Φ_α is a regular mapping from U^*_α onto U_α. If a point $\mathfrak{p}$ on U_α is not on V, then there is one and only one point $\mathfrak{p}^*$ on U^*_α such that $\Phi_\alpha(\mathfrak{p}^*) = \mathfrak{p}$, and Φ_α bi-regularly maps some open neighborhood of $\mathfrak{p}^*$ on U^*_α onto an open neighborhood of $\mathfrak{p}$ on U_α.

If $\mathfrak{p} \in U_\alpha \cap V$, then $\Phi_\alpha^{-1}(\mathfrak{p})$ is a non-singular subvariety on U^*_α and it is in bi-regular correspondence with a projective space of dimension $r-d-1$; in fact, $\Phi_\alpha^{-1}(\mathfrak{p}) = (\mathfrak{p}) \times P^*_\alpha$. Furthermore, $V^*_\alpha = \Phi_\alpha^{-1}(U_\alpha \cap V)$ is a non-singular subvariety of dimension $r-1$ on U^*_α and it is in bi-regular correspondence with the product variety $(U_\alpha \cap V) \times P^*_\alpha$. Let $U^*_{\alpha,h}$ denote the open set on U^*_α consisting of all points $\mathfrak{p}^*$ which satisfy $\zeta^\alpha_h(\mathfrak{p}^*) \neq 0$. Then the function x^α_h generates the ideal determined by V^*_α in the local ring of U^*_α at every point on $U^*_{\alpha,h}$.

We construct a bi-regular mapping $\eta_{\beta\alpha}$ from $\Phi_\alpha^{-1}(U_\alpha \cap U_\beta)$ onto $\Phi_\beta^{-1}(U_\alpha \cap U_\beta)$ for each $\alpha, \beta \in J^*$ as follows: for $\mathfrak{p}^* \in \Phi_\alpha^{-1}(U_\alpha \cap U_\beta)$, $\eta_{\beta\alpha}(\mathfrak{p}^*)$ is that point on $\Phi_\beta^{-1}(U_\alpha \cap U_\beta)$ such that

1) $\Phi_\beta(\eta_{\beta\alpha}(\mathfrak{p}^*)) = \Phi_\alpha(\mathfrak{p}^*) = \mathfrak{p} \in U_\alpha \cap U_\beta$;

2) $\rho_1 \zeta^\alpha_h(\mathfrak{p}^*) = \rho_2 \sum_{g=1}^{r-d} N^\beta{}_g{}^\alpha{}_h(\mathfrak{p}) \zeta^\beta_g(\eta_{\beta\alpha}(\mathfrak{p}^*)), \quad 1 \leqq h \leqq r-d.$

$\eta_{\beta\alpha}$ is a bi-regular mapping since $\det N^{\beta\alpha}$ is a unit in the local ring of X at every point $\mathfrak{p}$ on $U_\alpha \cap U_\beta \cap V$, as follows from our assumptions that $x^\alpha_1, \cdots, x^\alpha_{r-d}$ (resp. $x^\beta_1, \cdots, x^\beta_{r-d}$) generate the ideal determined by V in the local ring of X at every point on U_α (resp. U_β), and that V is a non-singular variety ($\det N^{\beta\alpha}$ is the determinant of the square matrix $N^{\beta\alpha}$ of degree $r-d$ whose entry in the g-th row and h-th column is $N^\beta{}_g{}^\alpha{}_h$.) Now from the equations

$$x^\alpha_h = \sum_{g=1}^{r-d} N^\beta{}_g{}^\alpha{}_h x^\beta_g, \qquad 1 \leqq h \leqq r-d,$$

$$x^\alpha_{h_1}\zeta^\alpha_{h_2} - x^\alpha_{h_2}\zeta^\alpha_{h_1} = 0, \qquad 1 \leqq h_1, h_2 \leqq r-d,$$

$$x^\beta_{h_1}\zeta^\beta_{h_2} - x^\beta_{h_2}\zeta^\beta_{h_1} = 0, \qquad 1 \leqq h_1, h_2 \leqq r-d,$$

it is easy to check the following assertions. The composite mapping $\eta_{\alpha\beta}\eta_{\beta\alpha}$ is the identity map of $\Phi_\alpha^{-1}(U_\alpha \cap U_\beta)$; $\eta_{\alpha\gamma}\eta_{\gamma\beta}\eta_{\beta\alpha}$ is the identity map of $\Phi_\alpha^{-1}(U_\alpha \cap U_\beta \cap U_\gamma)$; the restriction of Φ_α to $\Phi_\alpha^{-1}(U_\alpha \cap U_\beta)$ and $\Phi_\beta\eta_{\beta\alpha}$ are equal regular mappings from $\Phi_\alpha^{-1}(U_\alpha \cap U_\beta)$ onto $U_\alpha \cap U_\beta$.

Let U_0 denote the complement $X - V$ in X, and let Φ_0 denote the identity map of U_0. For each $\alpha \in J^*$, let $\eta_{0\alpha}$ denote the bi-regular mapping of $\Phi_\alpha^{-1}(U_0 \cap U_\alpha)$ onto $\Phi_0^{-1}(U_0 \cap U_\alpha) = U_0 \cap U_\alpha$ which is the restriction of Φ_α to $\Phi_\alpha^{-1}(U_0 \cap U_\alpha)$. U_0 and the U^*_α, for all $\alpha \in J^*$, can be merged into a single variety X^*. For let us identify the open set $\Phi_\alpha^{-1}(U_\alpha \cap U_\beta)$ on U^*_α with the open set $\Phi_\beta^{-1}(U_\alpha \cap U_\beta)$ on U^*_β for each $\alpha, \beta \in J^*$ according to the mapping $\eta_{\beta\alpha}$ and identify $\Phi_\alpha^{-1}(U_0 \cap U_\alpha)$ with $U_0 \cap U_\alpha$ according to $\eta_{0\alpha}$. The family $\{U^*_\alpha\}_{\alpha \in J^*}$ together with U_0 forms an open covering on X^*. X^* is a non-singular algebraic variety of dimension r and there is a unique regular mapping Φ from X^* onto X whose restriction to each U^*_α is Φ_α (the restriction of Φ to U_0 is Φ_0).

We have the non-singular subvariety V^* of dimension $r-1$ on X^* such that $U^*_\alpha \cap V^* = V^*_\alpha$ for each $\alpha \in J^*$, and $U_0 \cap V^*$ is empty. The restriction of Φ to V^* equips that variety with the structure of a projective fiber bundle whose base space is V and whose fiber is a projective space of dimension $r-d-1$. Let $\mathfrak{p}^* \in U^*_\alpha \cap U^*_\beta \cap V^*$. Then we have

$$\rho_1 \zeta^\alpha{}_h(\mathfrak{p}^*) = \rho_2 \sum_{g=1}^{r-d} N^\beta{}_g{}^\alpha{}_h(\mathfrak{p}^*) \zeta^\beta{}_g(\mathfrak{p}^*), \qquad 1 \leqq h \leqq r-d.$$

This permits us to identify V^* with the dual projective bundle $\mathfrak{B}(N(X;Y))$ of the sheaf $N(X;V)$ according to the rule that $\zeta^\alpha{}_1, \cdots, \zeta^\alpha{}_{r-d}$ are the homogeneous coordinates which are paired to the previous sections $z^\alpha{}_1, \cdots, z^\alpha{}_{r-d}$ with reference to the construction of $\mathfrak{B}(N(X;V))$. The restriction of Φ to V^* is then identical to the bundle projection of $\mathfrak{B}(N(X;V))$.

The mapping Φ is called an anti-monoidal transformation, and the variety V^* is called the anti-center of Φ. Φ is bi-regular if $d = r-1$.

§ 11. Let $\vartheta(V^*)$ be the divisor class on X^* of the divisor V^*. We recall the construction of the sheaf $\mathfrak{O}(-\vartheta(V^*))$. The restriction of $\mathfrak{O}(-\vartheta(V^*))$ to any $U^*_{\alpha,h}$ is a free sheaf of dimension one generated by a section $D[\alpha, h]$; its restriction to U_0 is generated by $D[0]$; on $U^*_{\alpha,h_0} \cap U^*_{\beta,h_1}$, we have the transition law

$$D[\alpha, h_0] = \Big(\sum_{h=1}^{r-d} N^\beta{}_h{}^\alpha{}_{h_0} \zeta^\beta{}_h / {}^\beta{}_{h_1}\Big) D[\beta, h_1],$$

and on $U_0 \cap U^*_{\alpha,h_0}$, we have

$$D[\alpha, h_0] = x^\alpha{}_{h_0} D[0].$$

Now we have

$$x^{\alpha}{}_{h_0} = (\sum_{h=1}^{r-d} N^{\beta}{}_{h}{}^{\alpha}{}_{h_0}\zeta^{\beta}{}_{h}/\zeta^{\beta}{}_{h_1})x^{\beta}{}_{h_1}$$

so that $\mathcal{O}(-\vartheta(V^*))$ is isomorphic to sheaf $\mathcal{L}(V^*)$ of germs of rational functions on X^* which are multiples of the divisor V^*. We have that

$$\Theta(\mathcal{L}(V^*)) = -\vartheta(V^*).$$

We obtain by inspection that the induced sheaf of $\mathcal{O}(-\vartheta(V^*))$ on the subvariety V^* is the basic sheaf $B(N(X;V))$ of $N(X;V)$; consequently, the basic divisor class $\Theta(N(X;V))$ is equal to the trace of $-\vartheta(V^*)$ on V^* (i.e., the reciprocal image of the divisor class $-\vartheta(V^*)$ with respect to the identity map of V^* in X^*).

Let Δ_X denote the diagonal on the product variety $X \times X$. Δ_X is nonsingular and it is in bi-regular correspondence with X in a specific way. The sheaf $N(X \times X;\Delta_X)$ is therefore intrinsically associated with the variety X; in fact, the reciprocal image of $N(X \times X;\Delta_X)$ with respect to the canonical mapping of X onto Δ_X is none other than the sheaf $\Omega^1{}_X$ of germs of differentials of degree one on X.

§ 12. The sheaf $\Phi\Omega^1{}_X$, which is the reciprocal image of $\Omega^1{}_X$ with respect to the mapping Φ of X^* into X, admits an isomorphism into the sheaf $\Omega^1{}_{X^*}$ of germs of differentials of degree one on X^* in a specific way. In fact, we shall establish the exact sequence of sheaves

$$0 \to \Phi\Omega^1{}_X \to \Omega^1{}_{X^*} \to \delta N \otimes B^{-1}(N) \to 0;$$

δN denotes the derived sheaf of $N(X;V)$ and $B^{-1}(N)$ denotes the reciprocal sheaf of the basic sheaf $B(N(X;V))$; both of these last sheaves are defined on V^* since we can identify V^* and $\mathcal{B}(N(X;V))$; their tensor product extended to a sheaf on X^* is the third term in the above exact sequence.

We can suppose that our covering of X has the property that for each $\alpha \in J^*$, there exist d regular functions $y^{\alpha}{}_1, \cdots, y^{\alpha}{}_d$ on U_α such that the r functions

$$x^{\alpha}{}_h - x^{\alpha}{}_h(\mathfrak{p}), \qquad 1 \leqq h \leqq r-d,$$

$$y^{\alpha}{}_i - y^{\alpha}{}_i(\mathfrak{p}), \qquad 1 \leqq i \leqq d,$$

generate the maximal prime ideal in the local ring $\mathcal{O}_{\mathfrak{p}}$ for every point $\mathfrak{p}$ on U_α; we can always refine the covering so as to secure this possibility. On the open set $U^*{}_{\alpha,h_0}$, we have that the r functions

$$y^{\alpha}_{i} - y^{\alpha}_{i}(\mathfrak{p}^*), \qquad 1 \leqq i \leqq d,$$

$$x^{\alpha}_{h_0} - x^{\alpha}_{h_0}(\mathfrak{p}^*),$$

$$\zeta^{\alpha}_{h}/\zeta^{\alpha}_{h} - \zeta^{\alpha}_{h}/\zeta^{\alpha}_{h_0}(\mathfrak{p}^*), \qquad 1 \leqq h \leqq r-d, h \neq h_0,$$

generate the maximal prime ideal of the local ring $\mathcal{O}_{\mathfrak{p}^*}$ of X^* at every point $\mathfrak{p}^*$ on U^*_{α,h_0}. (We use the same symbol for a rational function on X and its reciprocal image on X^*.) The restriction of $\Omega^1_{X^*}$ to U^*_{α,h_0} is the free sheaf of dimension r generated by the differentials

$$dy^{\alpha}_{i}, \qquad 1 \leqq i \leqq d,$$

$$dx^{\alpha}_{h_0},$$

$$d(\zeta^{\alpha}_{h}/\zeta^{\alpha}_{h_0}), \qquad 1 \leqq h \leqq r-d, h \neq h_0.$$

The restriction of $\Phi\Omega^1_X$ to $U^*_{\alpha,h}$ is a free sheaf which is the subsheaf of the restriction of $\Omega^1_{X^*}$ to U^*_{α,h_0} generated by

$$dy^{\alpha}_{i}, \qquad 1 \leqq i \leqq d,$$

$$dx^{\alpha}_{h}, \qquad 1 \leqq h \leqq r-d.$$

But we have on X^* that

$$x^{\alpha}_{h} = x^{\alpha}_{h_0}\zeta^{\alpha}_{h}/\zeta^{\alpha}_{h_0}, \qquad 1 \leqq h \leqq r-d,$$

which gives

$$dx^{\alpha}_{h} = \zeta^{\alpha}_{h}/\zeta^{\alpha}_{h_0}dx^{\alpha}_{h_0} + x^{\alpha}_{h_0}d(\zeta^{\alpha}_{h}/\zeta^{\alpha}_{h_0}), \qquad 1 \leqq h \leqq r-d, h \neq h_0;$$

consequently, the restriction of $\Phi\Omega^1_X$ to U^*_{α,h_0} is generated by the sections

$$dy^{\alpha}_{i}, \qquad 1 \leqq i \leqq d,$$

$$dx^{\alpha}_{h_0},$$

$$x^{\alpha}_{h_0}d(\zeta^{\alpha}_{h}/\zeta^{\alpha}_{h_0}) = dx^{\alpha}_{h} - (\zeta^{\alpha}_{h}/\zeta^{\alpha}_{h_0})\,dx^{\alpha}_{h_0}, \qquad 1 \leqq h \leqq r-d, h \neq h_0.$$

The restrictions of $\Phi\Omega^1_X$ and $\Omega^1_{X^*}$ to $X^* - V^*$ are equal since Φ bi-regularly maps $X^* - V^*$ onto $X - V$.

It is now evident that the quotient sheaf of $\Omega^1_{X^*}$ modulo $\Phi\Omega^1_X$ is the extension to X^* of a locally free sheaf of dimension $r-d-1$ defined on V^*. By inspection, we obtain that this quotient sheaf is the extension to X^* of the locally free sheaf $\delta N \otimes B^{-1}(N)$ defined on V^*; thus we have our exact sequence

$$0 \to \Phi\Omega^1_X \to \Omega^1_{X^*} \to \delta N \otimes B^{-1}(N) \to 0.$$

Consider the sheaf $(\Phi\Omega^1_X)'$ which is the induced sheaf of $\Phi\Omega^1_X$ on the

subvariety V^*; $(\Phi\Omega^1{}_X)'$ is identical with the reciprocal image of the induced sheaf of $\Omega^1{}_X$ on V with respect to the restriction of Φ to V^*, which mapping can be identified with the bundle projection π_N of $N(X;V)$. We have a specific isomorphism of $\pi_N N(X;V)$ into $(\Phi\Omega^1{}_X)'$ according to the rule that

$$\pi_N{}^{-1}z^\alpha{}_h \to (dx^\alpha{}_h)', \qquad 1 \leqq h \leqq r-d,$$

on each $U^*{}_\alpha \cap V^*$; the $\pi_N{}^{-1}z^\alpha{}_h$ are the reciprocal images of the previous sections $z^\alpha{}_h$ for N over U_α. Thus we have the exact sequence

$$0 \to \pi_N N(X;V) \to (\Phi\Omega^1{}_X)' \to \pi_N\Omega^1{}_V \to 0$$

of locally free sheaves defined on V^*. We view $\pi_N N(X;V)$ as subsheaf of $(\Phi\Omega^1{}_X)'$ so that δN is the subsheaf $(\Phi\Omega^1{}_X)'$ whose restriction to $U^*{}_{\alpha,h_0}$ is the free sheaf generated by the sections

$$(dx^\alpha{}_h)' - (\zeta^\alpha{}_h/\zeta^\alpha{}_{h_0})\,(dx^\alpha{}_{h_0})', \qquad 1 \leqq h \leqq r-d, h \neq h_0.$$

Let $(\Omega^1{}_{X^*})'$ be the induced sheaf of $\Omega^1{}_{X^*}$ on V^*. Then the induced homomorphism from $(\Phi\Omega^1{}_X)'$ to $(\Omega^1{}_{X^*})'$ is described on $U^*{}_{\alpha,h_0} \cap V^*$ according to

$$\begin{aligned} (dy^\alpha{}_i)' &\to (dy^\alpha{}_i)', & 1 \leqq i \leqq d,\\ (dx^\alpha{}_{h_0})' &\to (dx^\alpha{}_{h_0})', & \\ (dx^\alpha{}_h)' - (\zeta^\alpha{}_h/\zeta^\alpha{}_{h_0})\,(dx^\alpha{}_{h_0})' &\to 0, & 1 \leqq h \leqq r-d, h \neq h_0. \end{aligned}$$

The kernel of the induced hmomorphism is δN and the image is a locally free sheaf R of dimension $d+1$ defined on V^*; we have the following exact sequences of locally free sheaves defined on V^*

$$0 \to \delta N \to (\Phi\Omega^1{}_X)' \to R \to 0,$$

$$0 \to R \to (\Omega^1{}_{X^*})' \to \delta N \otimes B^{-1}(N) \to 0.$$

The induced homomorphism from $(\Phi\Omega^1{}_X)'$ to $(\Omega^1{}_{X^*})'$ maps $\pi_N N(X;V)$ onto a locally free sheaf defined on V^* which is isomorphic to $B(N)$. Thus we can view $B(N)$ as a subsheaf of R and we have the exact sequence

$$0 \to B(N) \to R \to \pi_N\Omega^1{}_V \to 0.$$

II. Locally Free Resolutions of Sheaves.

§ 13. Let X and V be as in § 9-§ 12. Let Q be a locally free sheaf defined on V, and consider the extension of Q to a sheaf defined on X whose stalks are the zero module over each point of X not on V (which sheaf we

continue to denote as Q). Given any (coherent, algebraic) sheaf S defined on X, then, according to one of Serre's fundamental theorems, it is possible to choose a locally free sheaf F and a homomorphism ψ of F onto S. This permits us to construct an exact sequence

$$F^t \xrightarrow{\psi^t} \cdots \to F^s \xrightarrow{\psi^s} F^{s-1} \to \cdots \to F^0 \xrightarrow{\psi^0} Q \to 0,$$

where F^s is a locally free sheaf defined on X for $0 \leqq s \leqq t$. If $s \geqq r-d-1$ (dim. $V = d$), then the sheaf $\operatorname{Ker}[\psi^s]$ (i.e., the kernel of ψ^s) is a locally free sheaf defined on X; and if $s \geqq r-d$, then the stalk $F^s{}_{\mathfrak{p}}$ at every point $\mathfrak{p}$ on X is the direct sum of the stalk $\operatorname{Ker}[\psi^s]_{\mathfrak{p}}$ and a free $\mathcal{O}_{\mathfrak{p}}$-module. These assertions follow from the assumption that X and V are non-singular in conjunction with well known arguments from cohomological algebra; they will be proved here in the course of establishing further results of this type.

We shall deal with a fixed exact sequence

$$(1) \qquad F^{r-d} \xrightarrow{\psi^{r-d}} \cdots \to F^s \xrightarrow{\psi^s} F^{s-1} \to \cdots \to F^0 \xrightarrow{\psi^0} Q \to 0$$

of length $r-d$, where F^s is a locally free sheaf defined on X for $0 \leqq s \leqq r-d$. We pass to the sheaves ΦQ, ΦF^s, $0 \leqq s \leqq r-d$, which are the reciprocal images of the sheaves Q, F^s, $0 \leqq s \leqq r-d$, with respect to the anti-monoidal transformation Φ from X^* onto X with anti-center V^*. We also have the reciprocal image homomorphisms

$$\Phi F^0 \xrightarrow{\Phi^{-1}(\psi^0)} \Phi Q,$$

$$\Phi F^s \xrightarrow{\Phi^{-1}(\psi^s)} \Phi F^{s-1}, \qquad 1 \leqq s \leqq r-d.$$

For notational convenience, set

$$Q_* = \Phi Q,$$
$$F^s{}_* = \Phi F^s,$$
$$\psi^s{}_* = \Phi^{-1}(\psi^s).$$

Thus we have arrived at the diagram of sheaves and homomorphisms on X^*

$$(2) \qquad F^{r-d}{}_* \xrightarrow{\psi^{r-d}{}_*} \cdots \to F^s{}_* \xrightarrow{\psi^s{}_*} F^{s-1}{}_* \to \cdots \to F^0{}_* \xrightarrow{\psi^0{}_*} Q_* \to 0;$$

but (2) *is not an exact sequence if* $r-d > 1$.

In the present chapter, we shall prove the following assertions.

1. The sheaf $\mathrm{Ker}[\psi^s{}_*]$ is a locally free sheaf defined on X^* for all $0 \leqq s \leqq r-d$; $\mathrm{Ker}[\psi^{r-d}]$ is a locally free sheaf defined on X and its reciprocal image with respect to Φ is $\mathrm{Ker}[\psi^{r-d}{}_*]$.

2. $\mathrm{Im}[\psi^s{}_*]$ is a locally free sheaf defined on X^* for $1 \leqq s \leqq r-d$; $\mathrm{Im}[\psi^1{}_*] = \mathrm{Ker}[\psi^0{}_*]$; $\mathrm{Im}[\psi^0{}_*] = Q_*$.

3. Let $Q^s{}_*$ denote the quotient sheaf of $\mathrm{Ker}[\psi^s{}_*]$ modulo $\mathrm{Im}[\psi^{s+1}{}_*]$ for $1 \leqq s \leqq r-d-1$; then $Q^s{}_*$ is isomorphic to $Q_* \otimes \wedge^s(\delta N)$, the tensor product of Q_* with the s-fold exterior product of the derived sheaf δN of $N(X;V)$; thus we hace the exact sequence

$$0 \to \mathrm{Im}[\psi^{s+1}{}_*] \to \mathrm{Ker}[\psi^s{}_*] \to Q_* \otimes \wedge^s(\delta N) \to 0$$

for $1 \leqq s \leqq r-d-1$.

There is another diagram of interest. Let Q', F'^s, $0 \leqq s \leqq r-d$, denote the induced sheaf of Q_*, $F^s{}_*$, $0 \leqq s \leqq r-d$, on the subvariety V^*; let ψ'^s denote the induced homomorphism of $\psi^s{}_*$ for $0 \leqq s \leqq r-d$. Then we have $Q' = Q_*$ and the diagram

$$F'^{r-d} \xrightarrow{\psi'^{r-d}} \cdots \to F'^s \xrightarrow{\psi'^s} F'^{s-1} \to \cdots \to F'^0 \xrightarrow{\psi'^0} Q_* \to 0, \tag{3}$$

where Q_*, F'^s, $0 \leqq s \leqq r-d$ are locally free sheaves defined on V^*. The diagram (3) *is not exact.* The following assertions are true.

1′. $\mathrm{Ker}[\psi'^s]$ is a locally free sheaf defined on V^* for all $0 \leqq s \leqq r-d$.

2′. $\mathrm{Im}[\psi'^s]$ is a locally free sheaf defined on V^* for all $0 \leqq s \leqq r-d$; $\mathrm{Im}[\psi'^0] = Q_*$; $\mathrm{Im}[\psi'^1] = \mathrm{Ker}[\psi'^0]$.

3′. The quotient sheaf of $\mathrm{Ker}[\psi'^s]$ modulo the subsheaf $\mathrm{Im}[\psi'^{s+1}]$ is for $1 \leqq s \leqq r-d$ isomorphic to the sheaf $Q_* \otimes \wedge^s(N^*)$, the tensor product of Q_* with the s-fold exterior product of the sheaf $N^* = \pi_N N(X;V)$ which is the reciprocal image of $N(X;V)$ with respect to π_N; thus we have the exact sequence of locally free sheaves on V^*

$$0 \to \mathrm{Im}[\psi'^{s+1}] \to \mathrm{Ker}[\psi'^s] \to Q_* \otimes \wedge^s(N^*) \to 0 \quad \text{for } 1 \leqq s \leqq r-d.$$

§ 14. The present § consists of formal algebra, and its aim is the proof of Lemma 1 which is the key to the proofs of the assertions of § 13.

Let $\boldsymbol{O}$ be a commutative ring with a multiplicative neutral element $1 \neq 0$ (eventually, $\boldsymbol{O}$ will be a local ring). We shall consider unitary $\boldsymbol{O}$-modules exclusively (i. e., 1 acts as the identity operator in any $\boldsymbol{O}$-module considered

here). Let $z_1, \cdots, z_t$ be elements in $\boldsymbol{O}$ which satisfy the Cartan-Eilenberg condition:

(C. E.) if for any $u \in \boldsymbol{O}$, we have that uz_i belongs to the ideal $(z_1, \cdots, z_{i-1})$ in $\boldsymbol{O}$ generated by $z_1, \cdots, z_{i-1}$, then u belongs to the ideal $(z_1, \cdots, z_{i-1})$ (in particular, for $i=1$, we require that z_1 is not a divisor of zero). We shall assume that the ideal $(z_1, \cdots, z_t)$ is not the unit ideal. Let $\boldsymbol{R}$ be the residue class ring of $\boldsymbol{O}$ modulo the ideal $(z_1, \cdots, z_t)$; then $\boldsymbol{R}$ has a multiplicative neutral element not zero; $\boldsymbol{R}$ carries the structure of an $\boldsymbol{O}$-module in a specific way and the residue class mapping μ_0 is an $\boldsymbol{O}$-homomorphism of $\boldsymbol{O}$ onto $\boldsymbol{R}$.

For $1 \leqq s \leqq t$, let M_s be the free $\boldsymbol{O}$-module of dimension $t!/s!(t-s)!$ generated by the symbols

$$[h_1, \cdots, h_s],$$

one such symbol for each strictly increasing sequence $1 \leqq h_1 < \cdots < h_s \leqq t$ of s integers chosen from the set $\{1, \cdots, t\}$. We construct an $\boldsymbol{O}$-homomorphism μ_s from M_s to M_{s-1} $(s > 1)$ according to the rule

$$\mu_s : [h_1, \cdots, h_s] \rightarrow \sum_{p=1}^{s} (-1)^{p-1} z_{h_p} [h_1, \cdots, \hat{h}_p, \cdots, h_s],$$

where $[h_1, \cdots, \hat{h}_p, \cdots, h_s]$ is that generator of M_{s-1} obtained by suppressing the p-th member h_p of the sequence $1 \leqq h_1 < \cdots < h_s \leqq t$. For $s=1$, we construct the homomorphism from M_1 to $\boldsymbol{O}$ according to

$$\mu_1 : [h] \rightarrow z_h, \qquad 1 \leqq h \leqq t.$$

It is a consequence of the Cartan-Eilenberg condition on $z_1, \cdots, z_t$ that

$$(1) \qquad 0 \rightarrow M_t \xrightarrow{\mu_t} \cdots \rightarrow M_s \xrightarrow{\mu_s} M_{s-1} \rightarrow \cdots \rightarrow M_1 \xrightarrow{\mu_1} \boldsymbol{O} \xrightarrow{\mu_0} \boldsymbol{R} \rightarrow 0$$

is an exact sequence of $\boldsymbol{O}$-modules; the proof is given in their book. The exact sequence (1) is called the Koszul resolution of $\boldsymbol{R}$ by free $\boldsymbol{O}$-modules.

Let L be a free $\boldsymbol{R}$-module of dimension q with the free basis $l_1, \cdots, l_q$. Then by the Koszul resolution of L with respect to the basis $l_1, \cdots, l_q$, we mean the direct sum of the Koszul resolution of $\boldsymbol{O}$ taken q times. Explicitly, $M_s{}^q$ $(1 \leqq s \leqq t)$ is a free $\boldsymbol{O}$-module of dimension $qt!/s!(t-s)!$ generated by the symbols

$$[h_1, \cdots, h_s; i];$$

one such symbol for each pair consisting of a strictly increasing sequence

$1 \leqq h_1 < \cdots < h_s \leqq t$ and an integer i, $1 \leqq i \leqq q$. The module $M^q{}_0$ is the free $\boldsymbol{O}$-module of dimension q generated by the symbols

$$[\,;i], \qquad\qquad 1 \leqq i \leqq q.$$

We construct an $\boldsymbol{O}$-homomorphism λ_s from $M^q{}_s$ to $M^q{}_{s-1}$ $(s > 1)$ according to

$$\lambda_s : [h_1, \cdots, h_s; i] \to \sum_{p=1}^{s} (-1)^{p-1} z_{h_p} [h_1, \cdots, \hat{h}_p, \cdots, h_s; i];$$

we define $\lambda_1 : M^q{}_1 \to M^q{}_0$ according to

$$\lambda_1 : [h; i] \to z_h[\,; i],$$

and $\lambda_0 : M^q{}_0 \to L$ according to

$$\lambda_0 : [\,; i] \to l_i.$$

The exact sequence

$$0 \to M^q{}_t \xrightarrow{\lambda_t} \cdots \to M^q{}_1 \xrightarrow{\lambda_1} M^q{}_0 \xrightarrow{\lambda_0} L \to 0 \tag{2}$$

is the Koszul resolution of L with respect to the basis $l_1, \cdots, l_q$.

We now assume that $\boldsymbol{O}$ is a local ring; recall that $z_1, \cdots, z_t$ are assumed to be non-units of $\boldsymbol{O}$. Consider an arbitrary resolution of L of length t. By this, we mean any exact sequence

$$E_t \xrightarrow{\phi_t} \cdots \to E_s \xrightarrow{\phi_s} E_{s-1} \to \cdots \to E_0 \xrightarrow{\phi_0} L \to 0, \tag{3}$$

where $E_0, \cdots, E_t$ are free $\boldsymbol{O}$-modules (all our modules are finitely generated). The following lemma gives the crucial property of the Koszul resolution; in particular, it gives that $\mathrm{Ker}[\phi_t]$ and $\mathrm{Ker}[\phi_{t-1}]$ are free $\boldsymbol{O}$-modules; more generally, the lemma gives that $\mathrm{Ker}[\phi_s]$ is isomorphic to the direct sum of $\mathrm{Ker}[\lambda_s]$ and a free $\boldsymbol{O}$-module for all $0 \leqq s \leqq t$.

LEMMA 1. *$\boldsymbol{O}$ is a (commutative) local ring with a multiplicative neutral element $1 \neq 0$; $z_1, \cdots, z_t$ is a set of non-units of $\boldsymbol{O}$ which satisfy the condition (C.E.); $\boldsymbol{R}$ is the residue class ring of $\boldsymbol{O}$ modulo the ideal $(z_1, \cdots, z_t)$; L is a free $\boldsymbol{R}$-module of dimension q; finally*

$$0 \to M^q{}_t \xrightarrow{\lambda_t} \cdots \to M^q{}_s \xrightarrow{\lambda_s} M^q{}_{s-1} \to \cdots \to M^q{}_0 \xrightarrow{\lambda_0} L \to 0 \tag{2}$$

is the Koszul resolution of L with respect to some free basis of the $\boldsymbol{R}$-module L (viewed as $\boldsymbol{O}$-module). Then given any exact sequence

$$E_t \xrightarrow{\phi_t} \cdots \to E_s \xrightarrow{\phi_s} E_{s-1} \to \cdots \to E_0 \xrightarrow{\phi_0} L \to 0, \tag{3}$$

where $E_0, \cdots, E_t$ *are free* $\boldsymbol{O}$*-modules, it is possible to construct homomorphisms*

$$\eta_s\colon M^q{}_s \to E_s, \qquad \xi_s\colon E_s \to M^q{}_s$$

for $0 \leqq s \leqq t$ *which satisfy:*

1.) $\xi_s\eta_s$ *is the identity map on* $M^q{}_s$;

2.) $\xi_{s-1}\phi_s = \lambda_s\xi_s$, $\eta_{s-1}\lambda_s = \phi_s\eta_s$ *(we shall agree that* ξ_{-1} *and* η_{-1} *denote the identity map of* L*).*

SUPPLEMENT TO LEMMA 1. *The module* $\mathrm{Ker}[\xi_s]$ *is a free* $\boldsymbol{O}$*-module;* $\mathrm{Ker}[\phi_s]$ *is the direct sum of* $\mathrm{Ker}[\xi_s]$ *and the image of* $\mathrm{Ker}[\lambda_s]$ *by* η_s, *which image is isomorphic to* $\mathrm{Ker}[\lambda_s]$.

The homomorphisms ξ_s and η_s are constructed by induction on s. It is convenient to cast part of the proof in the guise of several elementary lemmas; only one of these requires a demonstration.

LEMMA 2. *E, G are free modules of dimensions n and m over the local ring* $\boldsymbol{O}$; ψ *is a homomorphism from G onto E. Then given a free basis* $e_1, \cdots, e_n$ *for E, it is possible to choose a free basis* $g_1, \cdots, g_m$ *for G such that*

$$\psi(g_i) = e_i, \qquad 1 \leqq i \leqq n,$$

$$\psi(g_{n+u}) = 0, \qquad 1 \leqq n \leqq m - n;$$

or, without reference to bases, $\mathrm{Ker}[\psi]$ *is a free* $\boldsymbol{O}$*-module and there exists an isomorphism* ψ' *from E into G such that* $\psi\psi'$ *is the identity map of E; consequently, G is the direct sum of* $\mathrm{Ker}[\psi]$ *and* $\mathrm{Im}[\psi']$.

LEMMA 3. *E is a free module of dimension n over the local ring* $\boldsymbol{O}$ *and* $e_1, \cdots, e_n$ *is a free basis for E. Let* $e'_1, \cdots, e'_n$ *be elements of E such that for each i,* $1 \leqq i \leqq n$,

$$e'_i - e_i$$

is equal to a linear combination of $e_1, \cdots, e_n$ *whose coefficients are non-units of the local ring* $\boldsymbol{O}$. *Then* $e'_1, \cdots, e'_n$ *is a free basis for E.*

LEMMA 4. (This lemma reiterates the well known fact that a free module is a projective module, and it does not require the assumption that $\boldsymbol{O}$ be a local ring.) *E is a free* $\boldsymbol{O}$*-module; A, A' are* $\boldsymbol{O}$*-modules;* θ *is a homomirphism of A onto A';* π' *is a homomorphism of E into A'. Then there exists a homomorphism* π *of E into A such that* $\theta\pi$ *is equal to* π'.

LEMMA 5. (Retain the hypotheses and notations of Lemma 1.) *Let π' be a homomorphism of E_s onto the submodule* $\mathrm{Im}[\lambda_s]$ *of $M^q{}_{s-1}$; if $s=0$, then* $\mathrm{Im}[\lambda_0]=L$. *We can apply Lemma 4 since E_s is a free $\boldsymbol{O}$-module and λ_s maps $M^q{}_s$ onto* $\mathrm{Im}[\lambda_s]$, *which assures the existence of a homomorphism π of E_s into $M^q{}_s$ with the property $\lambda_s\pi=\pi'$. Then we claim that π maps E_s onto $M^q{}_s$.*

Proof. We have that $\mathrm{Im}[\lambda_s]$ is the submodule of $M^q{}_{s-1}$ generated by the elements

$$\sum_{p=1}^{s}(-1)^{p-1}z_{h_p}[h_1,\cdots,\hat{h}_p,\cdots,h_s],$$

one such element for each strictly increasing sequence $1\leqq h_1<\cdots<h_s\leqq t$ of length s and integer i, $1\leqq 0\leqq q$; (for $s=0$, we have $\mathrm{Im}[\lambda_0]=L$ is generated by $l_1,\cdots,l_q$). Since π' maps E_s onto $\mathrm{Im}[\lambda_s]$, we can choose elements

$$e'[h_1,\cdots,h_s;i], \qquad 1\leqq h_1<\cdots<h_s\leqq t, 1\leqq i\leqq q,$$

such that

$$\pi'(e'[h_1,\cdots,h_s;i])=\sum_{p=1}^{s}(-1)^{p-1}z_{h_p}[h_1,\cdots,\hat{h}_p,\cdots,h_s;i];$$

(for $s=0$, we choose $e'[i]$, $i\leqq 1\leqq q$, such that

$$\pi'(e'[i])=l_i).$$

We choose a homomorphism π of E_s into $M^q{}_s$ with the property $\lambda_s\pi=\pi'$. The elements

$$\pi(e'[h_1,\cdots,h_s;i])-[h_1,\cdots,h_s;i]$$

are all in the submodule $\mathrm{Ker}[\lambda_s]$ since

$$\lambda_s([h_1,\cdots,h_s;i])=\sum_{p=1}^{s}(-1)^{p-1}z_{h_p}[h_1,\cdots,\hat{h}_p,\cdots,h_s;i].$$

But $\mathrm{Ker}[\lambda_s]$ is generated by the elements

$$\sum_{p=1}^{s}(-1)^{p-1}z_{h_p}[h_1,\cdots,\hat{h}_p,\cdots,h_{s+1};i],$$

one such element for each strictly increasing sequence $1\leqq h_1<\cdots<h_{s+1}\leqq t$ of length $s+1$ and integer i, $1\leqq i\leqq q$. We can apply Lemma 3 since $z_1,\cdots,z_t$ are non-units of $\boldsymbol{O}$, and this proves that the elements

$$\pi(e'[h_1,\cdots,h_s;i]), \qquad 1\leqq h_1<\cdots<h_s\leqq t, 1\leqq i\leqq q,$$

generate $M^q{}_s$. Consequently, π maps E_s onto $M^q{}_s$.

Proof of Lemma 1. Assume that the homomorphisms $\xi_{-1}, \eta_{-1}, \cdots, \xi_{s-1}, \eta_{s-1}$ with the required properties have been constructed and $0 \leqq s \leqq t$; recall that ξ_{-1}, η_{-1} are both the identity map of L. We claim that the composite homomorphism $\xi_{s-1}\phi_s$ maps E_s onto $\mathrm{Im}[\lambda_s]$. This is obvious if $s=0$. If $s>0$, then for any element m of $\mathrm{Im}[\lambda_s]$, we have that $\eta_{s-1}(m)$ belongs to $\mathrm{Im}[\phi_s]$ since $\phi_{s-1}(\eta_{s-1}(m)) = \eta_{s-2}(\lambda_{s-1}(m)) = 0$ by our inductive assumption and the fact that $\mathrm{Im}[\phi_s] = \mathrm{Ker}[\phi_{s-1}]$, and $\mathrm{Im}[\lambda_s] = \mathrm{Ker}[\lambda_{s-1}]$; consequently, we can choose an element e in E_s such that $\phi_s(e) = \eta_{s-1}(m)$, which gives $\xi_{s-1}(\phi_s(e)) = m$; thus $\mathrm{Im}[\xi_{s-1}\phi_s]$ contains $\mathrm{Im}[\lambda_s]$. On the other hand, we have

$$\lambda_{s-1}\xi_{s-1}\phi_s = \xi_{s-2}\phi_{s-1}\phi_s = 0,$$

which proves that $\mathrm{Im}[\xi_{s-1}\phi_s]$ is contained in $\mathrm{Im}[\lambda_s]$. It follows from Lemma 4 (as applied in the statement of Lemma 5 with $\xi_{s-1}\phi_s = \pi'$) that we can choose a homomorphism ξ_s of E_s into $M^q{}_s$ with the property $\lambda_s\xi_s = \xi_{s-1}\phi_s$; and from Lemma 5, it follows that ξ_s maps E_s onto $M^q{}_s$. Since E_s, $M^q{}_s$ are free $\boldsymbol{O}$-modules and ξ_s maps E_s onto $M^q{}_s$, we apply Lemma 2 and obtain an isomorphism η_s of $M^q{}_s$ into E_s such that $\xi_s\eta_s$ is the identity map of $M^q{}_s$. It remains to check that $\phi_s\eta_s = \eta_{s-1}\lambda_s$. To this end, we observe that $\xi_{s-1}\phi_s\eta_s = \lambda_s\xi_s\eta_s = \lambda_s$ since $\xi_{s-1}\phi_s = \lambda_s\xi_s$ and $\xi_s\eta_s$ is the identity map of $M^q{}_s$; this leads to $\eta_{s-1}\xi_{s-1}\phi_s\eta_s = \eta_{s-1}\lambda_s$ and since the restriction of $\eta_{s-1}\xi_{s-1}$ to the submodule $\mathrm{Im}[\eta_{s-1}]$ is the identity map of that module, we get $\phi_s\eta_s = \eta_{s-1}\lambda_s$.

Applying Lemma 2 to the homomorphism ξ_s from E_s onto $M^q{}_s$, it follows that $\mathrm{Ker}[\xi_s]$ is a free $\boldsymbol{O}$-module. We have that ξ_s maps $\mathrm{Ker}[\phi_s]$ onto $\mathrm{Ker}[\lambda_s]$ (for we proved at the start of the proof of Lemma 1 that $\xi_{s-1}\phi_s$ maps E_s onto $\mathrm{Ker}[\lambda_{s-1}]$); thus we have that $\mathrm{Ker}[\phi_s]$ is the direct sum of $\mathrm{Ker}[\xi_s]$ and the image of $\mathrm{Ker}[\phi_s]$ by $\eta_s\xi_s$, which image is isomorphic to $\mathrm{Ker}[\lambda_s]$. This completes the proof of the supplement to Lemma 1.

§15. Consider the exact sequence of sheaves

$$(1) \qquad F^{r-d} \xrightarrow{\psi^{r-d}} \cdots \to F^0 \xrightarrow{\psi^0} Q \to 0$$

of §13. At a point $\mathfrak{p}$ on X, we have the exact sequence of stalks

$$(2) \qquad F^{r-d}{}_{\mathfrak{p}} \xrightarrow{\psi^{r-d}{}_{\mathfrak{p}}} \cdots \to F^0{}_{\mathfrak{p}} \xrightarrow{\psi^0{}_{\mathfrak{p}}} Q_{\mathfrak{p}} \to 0.$$

If $\mathfrak{p}$ is not on V, then $Q_{\mathfrak{p}}$ is the zero module and $\mathrm{Ker}[\psi^s]_{\mathfrak{p}}$ is a free $\boldsymbol{O}_{\mathfrak{p}}$-module for all $0 \leqq s \leqq r-d$, as follows from Lemma 2 since $\boldsymbol{O}_{\mathfrak{p}}$ is a local ring.

Assume that $\mathfrak{p} \in V$. Then there exist $r-d$ regular functions $x_1, \cdots, x_{r-d}$ in $\mathfrak{O}_{\mathfrak{p}}$ which generate the ideal determined by V in $\mathfrak{O}_{\mathfrak{p}}$, and since $\mathfrak{p}$ is a simple point on V, they can be extended to a basis of r elements for the maximal prime ideal of $\mathfrak{O}_{\mathfrak{p}}$; consequently, $x_1, \cdots, x_{r-d}$ satisfy the condition (C. E.) of § 14. The residue class ring of $\mathfrak{O}_{\mathfrak{p}}$ modulo the ideal $(x_1, \cdots, x_{r-d})$ is the local ring $\mathfrak{O}(V; \mathfrak{p})$ of V at $\mathfrak{p}$. The stalk $Q_{\mathfrak{p}}$ is free module of dimension q over $\mathfrak{O}(V; \mathfrak{p})$. Thus the structure of the stalks $\mathrm{Ker}[\psi^s]_{\mathfrak{p}}$ is known on the basis of Lemma 1 of § 14 and its supplement. In particular, we have that $\mathrm{Ker}[\psi^{r-d}]$ is a locally free sheaf defined on X.

Taking account of the properties of locally free sheaves and the structure of the stalks $\mathrm{Ker}[\psi^s]_{\mathfrak{p}}$ as revealed by Lemma 1 of § 14, we can suppose that our covering $\{U_\alpha\}_{\alpha \in J}$ has the following properties:

1) The restriction of Q to any $U_\alpha \cap V$ (with $\alpha \in J^*$) is a free sheaf generated by sections $l^\alpha{}_1, \cdots, l^\alpha{}_q$;

2) The restrictions of F^s, $1 \leqq s \leqq r-d$, to U_α (again $\alpha \in J^*$) is a free sheaf of dimension n_s generated by sections

$$\begin{aligned} &f^\alpha[h_1, \cdots, h_s; i], && 1 \leqq h_1 < \cdots < h_s \leqq r-d, 1 \leqq i \leqq q, \\ &f^\alpha{}_{t_s+j}, && 1 \leqq j \leqq u_s - t_s, \\ &f^\alpha{}_{u_s+k}, && 1 \leqq k \leqq n_s - u_s, \end{aligned}$$

(the restriction of F^0 to U_α is generated by

$$\begin{aligned} &f^\alpha[\,; i], && 1 \leqq i \leqq q, \\ &f_{q+k}, && 1 \leqq k \leqq n_0 - q), \end{aligned}$$

where for $s \geqq 1$, we have

$$t_s = q(r-d)\,!/s\,!(r-d-s)\,!$$

$$u_s - t_s = n_{s-1} - u_{s-1},$$

and $u_0 = t_0 = q$;)

3) The restriction of ψ^0 to U_α is described by

$$\begin{aligned} &f^\alpha[\,; i] \to l^\alpha{}_i, && 1 \leqq i \leqq q, \\ &f^\alpha{}_{q+k} \to 0, && 1 \leqq k \leqq n_0 - q, \end{aligned}$$

and the restriction of $\mathrm{Ker}[\psi^0]$ to U_α is generated by the sections

$$\begin{aligned} &x^\alpha{}_h f^\alpha[\,; i], && 1 \leqq h \leqq r-d, 1 \leqq i \leqq q, \\ &f^\alpha{}_{q+k}, && 1 \leqq k \leqq n_0 - q \end{aligned}$$

4) The restriction of ψ^s $(s \geqq 1)$ to U_α is described by

$$f^\alpha[h_1,\cdots,h_s;i] \to \sum_{p=1}^{s}(-1)^{p-1}x^\alpha{}_{h_p}f^\alpha[h_1,\cdots,\hat{h}_p,\cdots,h_s;i],$$

$$f^\alpha{}_{t_s+j} \to f^\alpha{}_{u_{s-1}+j}, \qquad 1 \leqq j \leqq u_s - t_s = n_{s-1} - u_{s-1},$$

$$f^\alpha{}_{u_s+k} \to 0, \qquad 1 \leqq k \leqq n_{s-1} - u_{s-1};$$

consequently, the restriction of $\mathrm{Ker}[\psi^s]$ to U_α is generated by the

$$\sum_{p=1}^{s+1}(-1)^{p-1}x^\alpha{}_{h_p}f^\alpha[h_1,\cdots,\hat{h}_p,\cdots,h_s;i],$$

$$1 \leqq h_1 < \cdots < h_s \leqq r-d, 1 \leqq i \leqq q,$$

$$f^\alpha{}_{u_s+k}, \qquad 1 \leqq k \leqq n_s - u_s.$$

The functions $x^\alpha{}_1,\cdots,x^\alpha{}_{r-d}$ are as in § 9. We shall agree that $f^\alpha[h_1,\cdots,h_s;i]$ is defined for arbitrary sequences $h_1,\cdots,h_s$ chosen from $\{1,\cdots,r-d\}$ but that it is strictly skew-symmetric in the h's.

The restriction of $Q_* = \Phi Q$ to any $U^*{}_\alpha \cap V^*$ is a free sheaf generated by sections

$$l_*{}^\alpha{}_i, \qquad 1 \leqq i \leqq q,$$

where $l_*{}^\alpha{}_i$ is the reciprocal image of the section $l^\alpha{}_i$ of Q over U_α; similarly, the restriction of $F^s{}_*$ to $U^*{}_\alpha$ is the free sheaf generated by the sections

$$f^\alpha{}_*[h_1,\cdots,h_s;i], \qquad 1 \leqq h_1 < \cdots < h_s \leqq r-d, 1 \leqq i \leqq q,$$

$$f^\alpha{}_{*t_s+j}, \qquad 1 \leqq j \leqq u_s - t_s,$$

$$f^\alpha{}_{*u_s+k}, \qquad 1 \leqq k \leqq n_s - u_s.$$

It follows that the restriction of $F^s{}_*$ to $U^*{}_{\alpha,h_0}$ is the free sheaf generated by the sections

$$f^\alpha{}_*[h_0,h_1,\cdots,h_{s-1};i], \qquad 1 \leqq h_1 < \cdots < h_{s-1} \leqq r-d, 1 \leqq i \leqq q,$$

where $h_p \neq h_0$ for $1 \leqq p \leqq s-1$, the sections

$$f^\alpha{}_*[h_1,\cdots,h_s;i] + \sum_{p=1}^{s}(-1)^p(\zeta^\alpha{}_{h_p}/\zeta^\alpha{}_{h_0})f^\alpha{}_*[h_0,h_1,\cdots,\hat{h}_p,\cdots,h_s;i]$$

for all $1 \leqq h_1 < \cdots < h_s \leqq r-d$, $1 \leqq i \leqq q$, and $h_p \neq h_0$ for $1 \leqq p \leqq s$, and the sections

$$f^\alpha{}_{*t_s+j}, \qquad 1 \leqq j \leqq u_s - t_s,$$

$$f^\alpha{}_{*u_s+k}, \qquad 1 \leqq k \leqq n_s - u_s.$$

On $U^*{}_{\alpha,h_0}$, we have that

$$x^\alpha{}_h = (\zeta^\alpha{}_h/\zeta^\alpha{}_{h_0})x^\alpha{}_{h_0} \qquad 1 \leqq h \leqq r-d, h \neq h_0;$$

consequently, the restriction of $\psi^s{}_*$ $(s \geqq 1)$ to $U^*{}_{a,h_0}$ is described by

$$f^a{}_*[h_0, h_1, \cdots, h_{s-1}; i] \to x^a{}_{h_0}(f^a{}_*[h_1, \cdots, h_{s-1}; i]$$
$$+ \sum_{p=1}^{s-1} (-1)^p (\zeta^a{}_{ph}/\zeta^a{}_{h_0}) f^a{}_*[h_0, \cdots, \hat{h}_p, \cdots, h_{s-1}; i])$$

for all $1 \leqq h_1 < \cdots < h_{s-1} \leqq r - d$, $h_p \neq h_0$, $1 \leqq i \leqq q$,

$$f^a{}_*[h_1, \cdots, h_s; i] + \sum_{p=1}^{s} (-1)^p (\zeta^a{}_{h_p}/\zeta^a{}_{h_0}) f^a{}_*[h_0, h_1, \cdots, \hat{h}_p, \cdots, h_s; i] \to 0$$

for all $1 \leqq h_1 < \cdots < h_s \leqq r - d$, $h_p \neq h_0$, $1 \leqq i \leqq q$,

$$f^a{}_{*t_s+j} \to f^a{}_{*u_{s-1}+j}, \qquad 1 \leqq j \leqq u_s - t_s = n_{s-1} - u_{s-1},$$
$$f^a{}_{*u_s+k} \to 0, \qquad 1 \leqq k \leqq n_s - u_s.$$

This proves that the restriction of $\mathrm{Ker}[\psi^s{}_*]$, $s \geqq 1$, to $U^*{}_{a,h_0}$ is a free sheaf generated by the sections

$$f^a{}_*[h_1, \cdots, h_s; i] + \sum_{p=1}^{s} (-1)^p (\zeta^a{}_{h_p}/\zeta^a{}_{h_0}) f^\beta{}_*[h_0, h_1, \cdots, \hat{h}_p, \cdots, h_s; i]$$

for $1 \leqq h_1 < \cdots < h_s \leqq r - d$, $h_p \neq h_0$, $1 \leqq i \leqq q$,

$$f^a{}_{*u_s+k}, \qquad 1 \leqq k \leqq n_s - u_s;$$

furthermore, the restriction of $\mathrm{Im}[\psi^{s+1}{}_*]$ to $U^*{}_{a,h_0}$ is a free sheaf since it is the subsheaf of the restriction of $\mathrm{Ker}[\psi^s{}_*]$ to $U^*{}_{a,h_0}$ generated by

$$x^a{}_{h_0}(f^a{}_*[h_1, \cdots, h_s; i]$$
$$+ \sum_{p=1}^{s} (-1)^p (\zeta^a{}_{h_p}/\zeta^a{}_{h_0}) f^a{}_*[h_0, h_1, \cdots, \hat{h}_p, \cdots, h_s; i])$$

for $1 \leqq h_1 < \cdots < h_s \leqq r - d$, $h_p \neq h_0$, $1 \leqq i \leqq q$,

$$f^a{}_{*u_s+k}, \qquad 1 \leqq k \leqq n_s - u_s.$$

The restriction of $\mathrm{Ker}[\psi^0{}_*]$ to $U^*{}_{a,h_0}$ is equal to the restriction of $\mathrm{Im}[\psi^1{}_*]$ to $U^*{}_{a,h_0}$ which is the subsheaf of the restriction of $F^0{}_*$ to $U^*{}_{a,h_0}$ generated by

$$x^a{}_{h_0} f^a{}_*[\,; i], \qquad 1 \leqq i \leqq q,$$
$$f^a{}_{*q+k}, \qquad 1 \leqq k \leqq n_0 - q.$$

We conclude therefore that $\mathrm{Ker}[\psi^s{}_*]$, resp. $\mathrm{Im}[\psi^s{}_*]$, is a locally free sheaf defined on X^* for all $0 \leqq s \leqq r - d$, resp. $1 \leqq s \leqq r - d$; $\mathrm{Ker}[\psi^0{}_*] = \mathrm{Im}[\psi^1{}_*]$; $\mathrm{Ker}[\psi^{r-d}{}_*] = \Phi\,\mathrm{Ker}[\psi^{r-d}]$. The residue class sheaf $Q^s{}_*$ of $\mathrm{Ker}[\psi^s{}_*]$ modulo $\mathrm{Im}[\psi^{s+1}{}_*]$ for $1 \leqq s \leqq r - d - 1$ is evidently the extension

to X^* of a locally free sheaf of dimension $q(r-d-1)!/s!(r-d-1-s)!$ defined on V^*. By a rather straightforward calculation, one constructs an isomorphism of $Q^s{}_*$ onto $Q_* \otimes \wedge^s(\delta N)$; thus we have the exact sequence

$$(3) \qquad 0 \to \mathrm{Im}[\psi^{s+1}{}_*] \to \mathrm{Ker}[\psi^s{}_*] \to Q_* \otimes \wedge^s(\delta N) \to 0$$

for $1 \leqq s \leqq r-d-1$. This completes our discussion of the assertions 1., 2., and 3. of § 13; the assertions 1'., 2'., 3'. follow rather easily from the calculations of this § but we shall have no need of them in the sequel.

III. The Dual Rational Transformation.

§ 16. The variety X and the sheaf E are as in § 3; G is a locally free sheaf of dimension m defined on X, and ψ is a homomorphism from G into E with the property that $\mathrm{Im}[\psi]$ is not the zero sheaf. We are going to construct a rational transformation $\mathfrak{B}(\psi)$ from $\mathfrak{B}(E)$ into $\mathfrak{B}(G)$; it is called the dual rational transformation of the sheaf homomorphism $\psi\colon G \to E$.

It is permissible to asume that the previous covering $\{U_\alpha\}_{\alpha \in J}$ is such that the restriction of G to any U_α is a free sheaf of dimension m generated by sections $g^\alpha{}_1, \cdots, g^\alpha{}_m$ of G over U_α. With reference to G, we have the transition laws

$$g^\alpha{}_q = \sum_{p=1}^{m} G^\beta{}_p{}^\alpha{}_q g^\beta{}_p, \qquad 1 \leqq q \leqq m,$$

on $U_\alpha \cap U_\beta$; the functions $G^\beta{}_p{}^\alpha{}_q$, $1 \leqq p, q \leqq m$ are regular on $U_\alpha \cap U_\beta$. The restriction of ψ to any U_α is described by

$$\psi\colon g^\alpha{}_p \to \sum_{i=1}^{n} \lambda^{\alpha,i}{}_p e^\alpha{}_i, \qquad 1 \leqq p \leqq m;$$

the $\lambda^{\alpha,i}{}_p$, $1 \leqq i \leqq n$, $1 \leqq p \leqq m$, are regular functions on U_α, and our assumption that $\mathrm{Im}[\psi]$ is not the zero sheaf means that for any $\alpha \in J$, not all of the functions $\lambda^{\alpha,i}{}_p$ are equal to the function zero.

$\pi_G{}^{-1}(U_\alpha)$, the portion of $\mathfrak{B}(G)$ over U_α, is a product variety $U_\alpha \times P'_\alpha$, where P'_α is a projective space of dimension $m-1$. Let $\omega^\alpha{}_1, \cdots, \omega^\alpha{}_m$ be the homogeneous coordinate system on P'_α which is paired with $g^\alpha{}_1, \cdots, g^\alpha{}_m$ over U_α with reference to $\mathfrak{B}(G)$; let $U'_{\alpha,p}$ denote the open subset on $U_\alpha \times P'_\alpha$ consisting of all points $\mathfrak{g}$ which satisfy $\omega^\alpha{}_p(\mathfrak{g}) \neq 0$; the family $\{U'_{\alpha,p}\}_{\alpha \in J, 1 \leqq p \leqq m}$ forms an open covering of $\mathfrak{B}(G)$. The restriction of the basic sheaf $B(G)$ to any $U'_{\alpha,p}$ is a free sheaf of dimension generated by the section $G[\alpha,p]$ of $B(G)$ over $U'_{\alpha,p}$. On $U'_{\alpha,p} \cap U'_{\beta,q}$, we have the transition law

$$G[\alpha,p] = \Big(\sum_{s=1}^{m} G^\beta{}_s{}^\alpha{}_p \omega^\beta{}_s/\omega^\beta{}_q\Big) G[\beta,q].$$

Consider the subset $Y(\psi)$ on the product variety $\mathcal{B}(E)\times\mathcal{B}(G)$ consisting of all points $(\mathfrak{e},\mathfrak{g})$ such that $\pi_E(\mathfrak{e})={}_G\pi(\mathfrak{g})$; $Y(\psi)$ is a non-singular subvariety on $\mathcal{B}(E)\times\mathcal{B}(G)$ of dimension $r+n+m-2$ ($\dim . X=r$). Let Σ_ψ denote the rational transformation from $Y(\psi)$ onto X which maps a point $(\mathfrak{e},\mathfrak{g})$ of $Y(\psi)$ onto the point $\mathfrak{p}=\pi_E(\mathfrak{e})=\pi_G(\mathfrak{g})$ of X; Σ_ψ is a regular mapping from $Y(\psi)$ onto X. The open set $\Sigma_\psi^{-1}(U_\alpha)$ on $Y(\psi)$, consisting of all points which Σ_ψ maps onto U_α, the the product variety $U_\alpha\times P_\alpha\times P'_\alpha$.

We shall say that a point $\mathfrak{e}$ on $U_\alpha\times P_\alpha$ satisfies the requirement (R) if:

(R) $\sum_{i=1}^{n}\lambda^{\alpha,i}{}_p(\mathfrak{p})\tau^\alpha{}_i(\mathfrak{e})\neq 0$ for at least one p, $1\leqq p\leqq m$; $\lambda^{\alpha,i}{}_p(\mathfrak{p})$ is the value of $\lambda^{\alpha,i}{}_p$ at the point $\mathfrak{p}=\pi_E(\mathfrak{e})$. The points of $U_\alpha\times P_\alpha$ which fail to satisfy (R) form a proper algebraic subset on $U_\alpha\times P_\alpha$ since $\mathrm{Im}[\psi]$ is not the sheaf zero. Let T_α be the closed (i. e., algebraic) set on $U_\alpha\times P_\alpha\times P'_\alpha$ consisting of all points such that

$$\rho_0\omega^\alpha{}_p(\mathfrak{g})=\rho_1\sum_{i=1}^{n}\lambda^{\alpha,i}{}_p(\mathfrak{p})\tau^\alpha{}_i(\mathfrak{e}),\qquad 1\leqq p\leqq m;$$

ρ_0, ρ_1 are arbitrary constants not both zero, and $\mathfrak{p}=\Sigma_\psi(\mathfrak{e},\mathfrak{g})$; T_α is a proper algebraic subset on $U_\alpha\times P_\alpha\times P'_\alpha$. If $\mathfrak{e}$ satisfies (R), then there is a unique point $\mathfrak{g}$ on $U_\alpha\times P'_\alpha$ such that $(\mathfrak{e},\mathfrak{g})$ is a point T_α. Consequently, there is a unique subvariety V_α of dimension $r+n-1$ on $U_\alpha\times P_\alpha\times P'_\alpha$ which is a maximal component of T_α, and with the property that V_α contains all points $(\mathfrak{e},\mathfrak{g})$ on T_α such that $\mathfrak{e}$ satisfies (R). Let η_α be that rational transformation from $U_\alpha\times P_\alpha$ into $U_\alpha\times P'_\alpha$ whose graph is V_α. It is easy to check that the restriction of η_α to $\pi_E^{-1}(U_\alpha\cap U_\beta)$ is equal to the restriction of η_β to $\pi_E^{-1}(U_\alpha\cap U_\beta)$; hence there is a unique rational transformation $\mathcal{B}(\psi)$ from $\mathcal{B}(E)$ into $\mathcal{B}(G)$ whose restriction to any $U_\alpha\times P_\alpha$ is equal to η_α. The requirement (R) is a sufficient condition for $\mathcal{B}(\psi)$ to be a regular at a point $\mathfrak{e}$; but in general, it is not a necessary condition.

Consider the subvariety $\mathcal{C}(\psi)$ on $Y(\psi)$ which is the graph of $\mathcal{B}(\psi)$; thet family $\{V_\alpha\}$ forms a covering of $\mathcal{C}(\psi)$ by open sets. Let $\psi_{;1}$ and $\psi_{;2}$ denote the projections from $\mathcal{C}(\psi)$ into $\mathcal{B}(E)$ and $\mathcal{B}(G)$ respectively; $\psi_{;1}B(E)$ and $\psi_{;2}B(G)$ are locally free sheaves of dimension one defined on $\mathcal{C}(\psi)$, for they are the reciprocal images of $B(E)$ and $B(G)$ with respect to $\psi_{;1}$ and $\psi_{;2}$ respectively. We shall exhibit a canonical homomorphism $B(\psi)$ of the sheaf $\psi_{;2}B(G)$ into $\psi_{;1}B(E)$; $B(\psi)$ is called the basic homomorphism associated with the sheaf homomorphism $\psi\colon G\to E$.

The restriction of $\psi_{;1}B(E)$, resp. $\psi_{;2}B(G)$, to $\psi_{;1}(U_{\alpha,i})$, resp. $\psi_{;2}(U'_{\alpha,p})$,

is a free sheaf of dimension one generated by the section $\psi_{;1}{}^{-1}E[\alpha, i]$, resp. $\psi_{;2}{}^{-1}G[\alpha, p]$, which is the reciprocal image of the section $E[\alpha, i]$, resp. $G[\alpha, p]$; (we consider here only those $U'_{\alpha,p}$ such that $\psi_{;2}{}^{-1}(U'_{\alpha,p})$ is non-empty). We have the transition laws

$$\psi_{;1}{}^{-1}E[\alpha, i] = \Big(\sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}(\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j})\Big)\psi_{;1}{}^{-1}E[\beta, j]$$

on $\psi_{;1}{}^{-1}(U_{\alpha,i} \cap U_{\beta,j})$, and

$$\psi_{;2}{}^{-1}G[\alpha, p] = \Big(\sum_{s=1}^{m} G^{\beta}{}_{s}{}^{\alpha}{}_{p}(\omega^{\beta}{}_{s}/\omega^{\beta}{}_{q})\Big)\psi_{;2}{}^{-1}G[\beta, q]$$

on $\psi_{;2}{}^{-1}(U'_{\alpha,p} \cap U'_{\beta,q})$; the coefficients in these transition laws are to be viewed as rational functions on $\mathcal{B}(\psi)$. For each triple $(\alpha; i, p)$, $\alpha \in J$, $1 \leqq i \leqq n$, $1 \leqq p \leqq m$, such that $\psi_{;1}{}^{-1}(U_{\alpha,i}) \cap \psi_{;2}{}^{-1}(U'_{\alpha,p})$ is non-empty, we set

$$U_{\alpha;i,p} = \psi_{;1}{}^{-1}(U_{\alpha,i}) \cap \psi_{;2}{}^{-1}(U'_{\alpha,p});$$

the family of all such admissible sets forms an open covering of $\mathcal{B}(\psi)$. The homomorphism $B(\psi)$ is constructed according to the rule that its restriction to any $U_{\alpha;i,p}$ is described by

$$\psi_{;2}{}^{-1}G[\alpha, p] \rightarrow \Big(\sum_{h=1}^{n} \lambda^{\alpha,h}{}_{p}(\tau^{\alpha}{}_{h}/\tau^{\alpha}{}_{i})\Big)\psi_{;1}{}^{-1}E[\alpha, i];$$

the coefficient is viewed as a regular function on $U_{\alpha;i,p}$. It is easy to verify that $B(\psi)$ has been constructed in a consistent way.

There is also a naturally determined locally free sheaf $S(\psi)$ of dimension one defined on $\mathcal{B}(\psi)$, and a naturally determined isomorphism of the product sheaf $(\psi_{;2}{}^{-1}B(G)) \otimes S(\psi)$ onto the sheaf $\psi_{;1}{}^{-1}B(E)$. The restriction of $S(\psi)$ to any $U_{\alpha;i,p}$ is a free sheaf of dimension one generated by a section $S[\alpha; i, p]$, and we have the transition law

$$S[\alpha; i, p] = \Big(\sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}(\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j})\Big)\Big(\sum_{s=1}^{m} G^{\beta}{}_{s}{}^{\alpha}{}_{p}(\omega^{\beta}{}_{s}/\omega^{\beta}{}_{p})\Big)^{-1} S[\beta; j, q]$$

on $U_{\alpha;i,p} \cap U_{\beta;j,q}$; the restriction of the isomorphism to any $U_{\alpha;i,p}$ is described by

$$(\psi_{;2}{}^{-1}G[\alpha, p]) \otimes S[\alpha; i, p] \rightarrow \psi_{;1}{}^{-1}E[\alpha; i, p].$$

Everything is consistent since we have

$$\Big(\sum_{h=1}^{n} \lambda^{\alpha,h}{}_{p}(\tau^{\alpha}{}_{h}/\tau^{\alpha}{}_{i})\Big) = \Big(\sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}(\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j})\Big)^{-1}\Big(\sum_{s=1}^{m} G^{\beta}{}_{s}{}^{\alpha}{}_{p}(\omega^{\beta}{}_{s}/\omega^{\beta}{}_{q})\Big)\Big(\sum_{h=1}^{n} \lambda^{\beta,h}{}_{q}(\tau^{\beta}{}_{h}/\tau^{\beta}{}_{j})\Big)$$

on $U_{\alpha;i,p} \cap U_{\beta;j,q}$.

§ **17.** Assume that ψ maps G onto E. It is permissible to suppose, on the basis of the definition of a coherent locally free sheaf together with Lemma 2 of § 14, that the families of generating section for E and G have been chosen such that the restriction of ψ to any U_α is described by

$$g^\alpha{}_i \to e^\alpha{}_i, \qquad 1 \leqq i \leqq n,$$

$$g^\alpha{}_{n+u} \to 0, \qquad 1 \leqq u \leqq m-n.$$

In this situation, $\mathfrak{B}(\psi)$ is a bi-regular mapping from $\mathfrak{B}(E)$ into $\mathfrak{B}(G)$; a point $\mathfrak{e}$ on $U_\alpha \times P_\alpha$ is mapped by $\mathfrak{B}(\psi)$ onto the point $\mathfrak{g}$ on $U_\alpha \times P'_\alpha$ such that

$$\pi_G(\mathfrak{g}) = \pi_E(\mathfrak{e}),$$

$$\omega^\alpha{}_i(\mathfrak{g}) = \tau^\alpha{}_i(\mathfrak{g}), \qquad 1 \leqq i \leqq n,$$

$$\omega^\alpha{}_{n+u}(\mathfrak{g}) = 0, \qquad 1 \leqq u \leqq m-n.$$

We view $\mathfrak{B}(E)$ as subvariety $\mathfrak{B}(G)$ by the device of identifying $\mathfrak{B}(E)$ with its image with respect to $\mathfrak{B}(\psi)$. If $p > n$, then $\mathfrak{B}(E) \cap U'_{\alpha,p}$ is empty. For any i, $1 \leqq i \leqq n$, the $m-n$ functions

$$\omega^\alpha{}_{n+u}/\omega^\alpha{}_i, \qquad 1 \leqq u \leqq m-n,$$

generate the ideal determined by $\mathfrak{B}(E)$ in the local ring of $\mathfrak{B}(G)$ at every point on $U'_{\alpha,i}$.

We have that

$$G^{\beta}{}_i{}^\alpha{}_j = E^{\beta}{}_i{}^\alpha{}_j, \qquad 1 \leqq i, j \leqq n,$$

$$G^{\beta}{}_p{}^\alpha{}_q = 0, \qquad 1 \leqq p \leqq n, n+1 \leqq q \leqq m-n;$$

which is to say, we have the transition laws

$$g^\alpha{}_j = \sum_{i=1}^{n} E^{\beta}{}_i{}^\alpha{}_j g^\beta{}_i + \sum_{u=1}^{m-n} G^{\beta}{}_{n+u}{}^\alpha{}_j g^\beta{}_{n+u}$$

for $1 \leqq j \leqq n$,

$$g^\alpha{}_{n+v} = \sum_{u=1}^{m-n} G^{\beta}{}_{n+u}{}^\alpha{}_{n+v} g^\beta{}_{n+u}$$

for $1 \leqq v \leqq m-n$. It is an immediate conseqence of these transition laws, together with the fact that the functions

$$\omega^\alpha{}_{n+u}/\omega^\alpha{}_i, \qquad 1 \leqq u \leqq m-n,$$

generate the ideal determined by $\mathfrak{B}(E)$ on $U'_{\alpha,i}$, that the induced sheaf of $B(G)$ on $\mathfrak{B}(E)$ is equal to $B(E)$.

Thus, under the assumption that ψ maps G onto E, we have that $\mathfrak{B}(\psi)$

is a bi-regular mapping of $\mathfrak{B}(E)$ onto a subvariety on $\mathfrak{B}(G)$; if we identify $\mathfrak{B}(E)$ with that subvariety, then the restriction of π_G to $\mathfrak{B}(E)$ is equal to π_E, and the induced sheaf of $B(G)$ on $\mathfrak{B}(E)$ is equal to $B(E)$; in particular, the trace of the divisor class $\Theta(G)$ on $\mathfrak{B}(E)$ is equal to $\Theta(E)$.

Suppose that $m > n$. The sheaf $\mathrm{Ker}[\psi]$ is a locally free sheaf of dimension $m-n$ defined on X; the restriction of $\mathrm{Ker}[\psi]$ to any U_α is the free sheaf generated by the sections $g^\alpha{}_{n+1},\cdots,g^\alpha{}_m$, and we have the transition laws

$$g^\alpha{}_{n+v}=\sum_{u=1}^{m-n}H^\beta{}_u{}^\alpha{}_v g^\beta{}_{n+u}, \qquad 1\leqq v\leqq m-n,$$

with

$$H^\beta{}_u{}^\alpha{}_v=G^\beta{}_{n+u}{}^\alpha{}_{n+v}, \qquad 1\leqq u,v\leqq m-n.$$

We construct a locally free sheaf H of dimension $m-n$ defined on X as follows: the restriction of H to U_α is a free sheaf generated by sections $h^\alpha{}_1,\cdots,h^\alpha{}_{m-n}$, and we have the transition laws

$$h^\alpha{}_v=\sum_{u=1}^{m-n}H^\beta{}_u{}^\alpha{}_v h^\beta{}_u, \qquad 1\leqq v\leqq m-n.$$

We construct an isomorphism θ from H into G according to the rule: the restriction of θ to U_α is described by

$$\theta\colon h^\alpha{}_u\to g^\alpha{}_{n+u}, \qquad 1\leqq u\leqq m-n;$$

the sheaf $\mathrm{Im}[\theta]$ is equal to $\mathrm{Ker}[\psi]$, but we must emphasize that the range of θ is the sheaf G; consequently, there is the exact sequence

$$0\to H\xrightarrow{\ \theta\ }G\xrightarrow{\ \psi\ }E\to 0$$

of locally free sheaves defined on X.

We shall examine the constructions of § 16 with refernce to the homomorphism $\theta\colon H\to G$. $\pi_H{}^{-1}(U_\alpha)$, the portion of $\mathfrak{B}(H)$ over U_α, is a product variety $U_\alpha\times P''_\alpha$; P''_α is a projective space of dimension $m-n-1$ with a homogeneous coordinate system $\eta^\alpha{}_1,\cdots,\eta^\alpha{}_{m-n}$ which is paired with the sections $h^\alpha{}_1,\cdots,h^\alpha{}_{m-n}$ over U_α with reference to the construction of $\mathfrak{B}(H)$. $U''_{\alpha,u}$, $1\leqq u\leqq m-n$, is the open set on $U_\alpha\times P''_\alpha$ consisting of all points $\mathfrak{h}$ such that $\eta^\alpha{}_u(\mathfrak{h})\neq 0$; the restriction of the basic sheaf $B(H)$ to $U''_{\alpha,u}$ is a free sheaf of dimension one generated by the section $H[\alpha,u]$; we have the transition law

$$H[\alpha,u]=\Big(\sum_{s=1}^{m-n}H^\beta{}_s{}^\alpha{}_u(\eta^\beta{}_s/\eta^\beta{}_v)\Big)H[\beta,v]$$

on $U''_{\alpha,u}\cap U''_{\alpha,v}$.

$Y(\theta)$ is the non-singular subvariety on $\mathcal{B}(G)\times\mathcal{B}(H)$ consisting of all points $(\mathfrak{g},\mathfrak{h})$ such that

$$\pi_G(\mathfrak{g})=\pi_H(\mathfrak{h});$$

Σ_θ is the projection from $Y(\theta)$ onto X which maps a point $(\mathfrak{g},\mathfrak{h})$ onto the point $\pi_G(\mathfrak{g})=\pi_H(\mathfrak{h})=\mathfrak{p}$ on X; $\Sigma_\theta^{-1}(U_\alpha)$ is the product variety $U_\alpha\times P'_\alpha\times P''_\alpha$. Let $(\mathfrak{g},\mathfrak{h})$ be a point on $\mathcal{G}(\theta)\cap\Sigma_\theta^{-1}(U_\alpha)$, and suppose that $\eta^\alpha{}_{u_0}(\mathfrak{h})\neq 0$, $\omega^\alpha{}_{p_0}(\mathfrak{g})\neq 0$. We shall prove that $(\mathfrak{g},\mathfrak{h})$ is a simple point on $\mathcal{G}(\theta)$.

Case I: $p_0>n$. The $m-n-1$ functions

$$\omega^\alpha{}_{n+u}/\omega^\alpha{}_{p_0}-\eta^\alpha{}_u/\eta^\alpha{}_{n_0}, \qquad u\neq p_0-n, 1\leqq n\leqq m-n,$$

generate the ideal determined by $\mathcal{G}(\theta)$ in the local ring of $Y(\theta)$ at $(\mathfrak{g},\mathfrak{h})$; visibly, $(\mathfrak{g},\mathfrak{h})$ is a simple point on $\mathcal{G}(\theta)$ and $\theta_{;1}$, the projection from $\mathcal{G}(\theta)$ onto $\mathcal{B}(G)$, bi-regularly maps some open neighborhood of $(\mathfrak{g},\mathfrak{h})$ on $\mathcal{G}(\theta)$ onto an open neighborhood of $\mathfrak{g}$ on $\mathcal{B}(G)$; in fact, the restriction of $\theta_{;1}$ to $\theta_{;1}{}^{-1}(U'_{\alpha,p_0})$ is a bi-regular map of that open set onto U'_{α,p_0}.

Case II: $\omega^\alpha{}_p(\mathfrak{g})=0$ for all $n<p\leqq m$, so that $\omega^\alpha{}_{i_0}(\mathfrak{g})\neq 0$ for some i_0, $1\leqq i_0\leqq n$. In this case, the $m-n-1$ functions

$$\omega^\alpha{}_{n+u}/\omega^\alpha{}_{i_0}-(\eta^\alpha{}_u/\eta^\alpha{}_{u_0})\,\omega^\alpha{}_{n+u_0}/\omega^\alpha{}_{i_0}, \qquad u\neq u_0, 1\leqq u\leqq m-n,$$

generate the ideal determined by $\mathcal{G}(\theta)$ in the local ring at $Y(\theta)$ at $(\mathfrak{g},\mathfrak{h})$; visibly, $(\mathfrak{g},\mathfrak{h})$ is a simple point on $\mathcal{G}(\theta)$; moreover, the variety $\theta_{;1}{}^{-1}(U'_{\alpha,i_0})$ is obtained from monoidal transformation of U'_{α,i_0} centered on the subvariety $\mathcal{B}(E)\cap U'_{\alpha,i_0}$, since the ideal determined by that subvariety is generated by the functions

$$\omega^\alpha{}_{n+u}/\omega^\alpha{}_{i_0}, \qquad 1\leqq u\leqq m-n,$$

in the local ring at every point on U'_{α,i_0}.

Thus the projection $\theta_{;1}$ is an anti-monoidal transformation from $\mathcal{G}(\theta)$ onto $\mathcal{B}(G)$; the center is the subvariety $\mathcal{B}(E)$ on $\mathcal{B}(G)$. Let $U'_{\alpha;p_0,u_0}$, $\alpha\in J$, $1\leqq p_0\leqq m$, $1\leqq u_0\leqq m-n$, denote the open subset on $\mathcal{G}(\theta)$ consisting of all points $(\mathfrak{g},\mathfrak{h})$ on $\mathcal{G}(\theta)\cap\Sigma_\theta^{-1}(U_\alpha)$ which satisfy

$$\omega^\alpha{}_{p_0}(\mathfrak{g})\neq 0, \qquad \eta^\alpha{}_{n_0}(\mathfrak{h})\neq 0;$$

let $\mathfrak{N}_\theta$ denote the anti-center of $\theta_{;1}$. Then $\mathfrak{N}_\theta\cap U'_{\alpha;p_0,u_0}$ is empty for $n<p_0\leqq m$, and for $p_0=i_0$, $i_0\leqq n$, the function

$$\omega^\alpha{}_{n+u_0}/\omega^\alpha{}_{i_0}$$

generates the ideal determined by $\mathfrak{N}_\theta$ in the local ring of $\mathcal{B}(\theta)$ at every point on $U'_{\alpha;p_0,u_0}$. From the previous discussion on monoidal transformations, we know that the restriction of $\theta_{;1}$ to $\mathfrak{N}_\theta$ equips that variety with the sttructure of a projecive fiber bundle whose base space is $\mathfrak{B}(E)$ and whose fiber is a projective space of dimension $m-n-1$; as such, $\mathfrak{N}_\theta$ is the dual projective bundle of the sheaf $N(\mathfrak{B}(G), \mathfrak{B}(E))$ of germs of covariant normal vectors fields to $\mathfrak{B}(E)$ in $\mathfrak{B}(G)$; the restriction of $\theta_{;1}$ to $\mathfrak{N}_\theta$ is equal to the bundle projection π_N ($\pi_N = \pi_{N(\mathfrak{B}(G);\mathfrak{B}(E))}$) of $\mathfrak{N}_\theta$ onto $\mathfrak{B}(E)$.

We have that

$$\omega^\alpha{}_{n+u}/\omega^\alpha{}_{i_0} = \left(\sum_{t=1}^{m-n} H^\beta{}_t{}^\alpha{}_u\,(\omega^\beta{}_{n+t}/\omega^\beta{}_{i_1})\right)\left(\sum_{s=1}^{m} G^\beta{}_s{}^\alpha{}_{i_0}(\omega^\beta{}_s/\omega^\beta{}_{i_1})\right)^{-1}$$

on $U'_{\alpha,i_0} \cap U'_{\beta,i_1}$ for all $1 \leqq u \leqq m-n$, where i_0, i_1 are fixed and $\mathrm{v} \leqq i_0, i_1 \leqq n$. Consequently, the restriction of $N(\mathfrak{B}(G); \mathfrak{B}(E))$ to any $U_{\alpha,i}$ $(1 \leqq i \leqq n)$ is a free sheaf of dimension $r-n$ generated by sections

$$z^{\alpha,i}{}_1, \cdots, z^{\alpha,i}{}_1,$$

and we have the transition laws

$$z^{\alpha,i}{}_u = \left(\sum_{s=1}^{n} E^\beta{}_s{}^\alpha{}_i(\omega^\beta{}_s/\omega^\beta{}_j)\right)^{-1} \sum_{t=1}^{m-n} H^\beta{}_t{}^\alpha{}_u z^{\beta,j}{}_t$$

on $U_{\alpha,i} \cap U_{\beta,j}$ for all $1 \leqq u \leqq m-n$, since we have

$$E^\beta{}_s{}^\alpha{}_i = G^\beta{}_s{}^\alpha{}_i \qquad 1 \leqq i, s \leqq n,$$

and $\omega^\beta{}_s/\omega_{\beta j}$ is the rational function zero on $U_{\alpha,i} = \mathfrak{B}(E) \cap U'_{\alpha,i}$ for $s > n$. By inspection, we obtain

$$N(\mathfrak{B}(G); \mathfrak{B}(E)) = B^{-1}(E) \otimes \pi_E H;$$

which is to say, the right hand side is the product sheaf of $B^{-1}(E)$ (where $B^{-1}(E) \otimes B(E) = \mathcal{O}_{\mathfrak{B}(E)}$) with the reciprocal image $\pi_E H$ of the sheaf H with respect to the mapping $\pi_E \colon \mathfrak{B}(E) \to X$.

Appealing to the constructions of § 16, we have the locally free sheaf $S(\theta)$ of dimension one defined on $\mathcal{B}(\theta)$ with the property

$$\theta_{;2}B(H) \otimes S(\theta) = \theta_{;1}B(G);$$

the sheaf $\theta_{;1}B(G)$, resp. $\theta_{;2}B(H)$, is the reciprocal image of $B(G)$, resp. $B(H)$, with respect to the mapping $\theta_{;1}$, resp. $\theta_{;2}$. The basic class $\Theta(S(\theta))$ of $S(\theta)$ is the divisor class of the divisor $\mathfrak{N}_\theta$ on $\mathcal{B}(\theta)$; this follows from the fact that

$$\omega^\alpha{}_{n+u}/\omega^\alpha{}_i = \left(\sum_{t=1}^{m-n} H^\beta{}_t{}^\alpha{}_u(\eta^\beta{}_t/\eta^\beta{}_v)\right)\left(\sum_{s=1}^{m} G^\beta{}_s{}^\alpha{}_i\omega^\beta{}_s/\omega^\beta{}_j\right)^{-1}\omega^\beta{}_{n+v}/\omega^\beta{}_j$$

on $U'_{\alpha;i,u} \cap U'_{\beta;j,v}$ for fixed $i, j, u, v, 1 \leqq i, j \leqq n, 1 \leqq u, v \leqq m-n$, and the transition law

$$S[\alpha;i,u] = (\sum_{t=1}^{m-n} H^{\beta}{}_{t}{}^{\alpha}{}_{n}(\eta^{\beta}{}_{t}/\eta^{\beta}{}_{v}))^{-1}(\sum_{s=1}^{m} G^{\beta}{}_{s}{}^{\alpha}{}_{i}\omega^{\beta}{}_{s}/\omega^{\beta}{}_{j})S[\beta;j,v]$$

on $U'_{\alpha;i,u} \cap U'_{\beta;j,v}$. By inspection, we obtain that the induced sheaf of $S(\theta)$ on the subvariety $\mathfrak{N}_{\theta}$ is the sheaf $B^{-1}(N(\mathfrak{B}(G);\mathfrak{B}(E)))$. We have arrived at the important formula

$$\theta_{;2}{}^{*}(\Theta(H)) + \Theta(S(\theta)) = \theta_{;1}{}^{*}(\Theta(G));$$

$\theta_{;1}{}^{*}(\Theta(G))$, resp. $\theta_{;2}{}^{*}(\Theta(H))$, is the reciprocal image of the basic divisor class $\Theta(G)$, resp. $\Theta(H)$, with respect to the projection $\theta_{;1}$, resp. $\theta_{;2}$, and $\Theta(S(\theta))$ is the divisor class of the anti-center $\mathfrak{N}_{\theta}$ of $\theta_{;1}$.

We shall prove that the projection $\theta_{;2}$ equips $\mathfrak{G}(\theta)$ with the structure of a projective fiber bundle whose base space is $\mathfrak{B}(H)$; as such $\mathfrak{G}(\theta)$ is the dual projective bundle of a certain locally free sheaf $R(\theta)$ of dimension $n+1$ defined on $\mathfrak{B}(H)$. Let $\mathfrak{h}$ be a point on U''_{α,u_0}. $\theta_{;2}{}^{-1}(\mathfrak{h})$ consists of all points $(\mathfrak{g},\mathfrak{h})$ such that $\mathfrak{g} \in U_{\alpha} \times P'_{\alpha}$ satisfies

(a) $\pi_G(\mathfrak{g}) = \pi_H(\mathfrak{h})$

(b) $\omega^{\alpha}{}_{n+u}(\mathfrak{g}) - (\eta^{\alpha}{}_{u}/\eta^{\alpha}{}_{u_0}(\mathfrak{h}))\omega^{\alpha}{}_{n+u_0}(\mathfrak{g}) = 0, \quad u \neq u_0, 1 \leqq u \leqq m-n;$

consequently, $\theta_{;2}{}^{-1}(U''_{\alpha,u_0})$ is the product variety of U''_{α,u_0} with a projective subspace of dimension n on P'_{α}; this subspace is described by the equations

$$\omega^{\alpha}{}_{n+u} - (\eta^{\alpha}{}_{u}/\eta^{\alpha}{}_{u_0})\omega^{\alpha}{}_{n+u_0} = 0, \qquad u \neq u_0, 1 \leqq u \leqq m-n.$$

If the point $(\mathfrak{g},\mathfrak{h})$ belongs to $\theta_{;2}{}^{-1}(U''_{\alpha,u_0}) \cap \theta_{;2}{}^{-1}(U''_{\beta,v_0})$, then we have (where $\mathfrak{p} = \pi_H(\mathfrak{h}) = \pi_G(\mathfrak{g})$)

$$\omega^{\alpha}{}_{i}(\mathfrak{g}) = \rho(\sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i}(\mathfrak{p})\omega^{\beta}{}_{h}(\mathfrak{g}) + (\sum_{t=1}^{m-n} G^{\beta}{}_{n+t}{}^{\alpha}{}_{i}(\mathfrak{p})\eta^{\beta}{}_{t}/\eta^{\beta}{}_{v_0}(\mathfrak{h}))\omega^{\beta}{}_{n+v_0}(\mathfrak{g})),$$

for all $1 \leqq i \leqq n$, and

$$\omega^{\alpha}{}_{n+u_0}(\mathfrak{g}) = \rho(\sum_{t=1}^{m-n} H^{\beta}{}_{t}{}^{\alpha}{}_{u_0}(\mathfrak{p})\eta^{\beta}{}_{t}/\eta^{\beta}{}_{v_0}(\mathfrak{h}))\omega^{\beta}{}_{n+v_0}(\mathfrak{g});$$

this proves that $\theta_{;2}$ equips $\mathfrak{G}(\theta)$ with the structure of a projective fiber bundle.

$R(\theta)$ is a locally free sheaf of dimension $n+1$ defined on $\mathfrak{B}(H)$ as follows: the restriction of $R(\theta)$ to U''_{α,u_0} $(\alpha \in J, 1 \leqq u_0 \leqq m-n)$ is a free sheaf of dimension $n+1$ generated by sections

$$r_1[\alpha,u_0], \cdots, r_n[\alpha,u_0], r_{n+1}[\alpha,u_0]$$

on $U''_{\alpha,u_0} \cap U''_{\beta,v_0}$; there is the transition law

$$r_i[\alpha, u_0] = \sum_{h=1}^{n} E^{\beta}{}_{h}{}^{\alpha}{}_{i} r_h[\beta, v_0] + \left(\sum_{t=1}^{m-n} G^{\beta}{}_{n+t}{}^{\alpha}{}_{i}(\eta^{\beta}{}_{t}/\eta^{\beta}{}_{v_0})\right) r_{n+1}[\beta, v_0]$$

for $1 \leqq i \leqq n$, and

$$r_{n+1}[\alpha, u_0] = \left(\sum_{t=1}^{m-n} H^{\beta}{}_{t}{}^{\alpha}{}_{u_0}(\eta^{\beta}{}_{t}/\eta^{\beta}{}_{v_0})\right) r_{n+1}[\beta, v_0].$$

We identify the dual projective bundle of the sheaf $R(\theta)$ with $\mathcal{B}(\theta)$ according to the rule that the homogeneous coordinates

$$\omega^{\alpha}{}_{1}, \cdots, \omega^{\alpha}{}_{n}, \omega^{\alpha}{}_{n+u_0}$$

are paired with the sections

$$r_1[\alpha, u_0], \cdots, r_n[\alpha, u_0], r_{n+1}[\alpha, u_0]$$

over U''_{α,u_0} with reference to the construction of $\mathcal{B}(R(\theta))$; the projection $\theta_{;2}$ is then equal to the bundle projection $\pi_{R(\theta)}$. There is an evident isomorphism of $B(H)$ into $R(\theta)$; the restriction of this isomorphism to any U''_{α,u_0} is described by

$$H[\alpha, u_0] \rightarrow r_{n+1}[\alpha, u_0].$$

It follows by inspection that the quotient sheaf of $R(\theta)$ modulo the image of $B(H)$ is isomorphic to the sheaf $\pi_H E$, the reciprocal image of E with respect to the mapping π_H. Thus we have the exact sequence

$$0 \rightarrow B(H) \rightarrow R(\theta) \rightarrow \pi_H E \rightarrow 0$$

of locally free sheaves defined on $\mathcal{B}(H)$.

The homomrophism from $R(\theta)$ onto $\pi_H E$ permits us to identity $\mathcal{B}(\pi_H E)$ with a subvariety on $\mathcal{B}(\theta)$ in such a fashion that the restriction of $\theta_{;2}$ to $\mathcal{B}(\pi_H E)$. More precisely, we can assert that $\mathcal{B}(\pi_H E)$, as a subvariety on $\mathcal{B}(\theta)$, is equal to the previously introduced variety $\mathfrak{N}_{\theta}$. For the restriction to U''_{α,u_0} of our homomorphism from $R(\theta)$ onto $\pi_H E$ is described by

$$r_i[\alpha, u_0] \rightarrow \pi_H{}^{-1} e^{\alpha}{}_{i}, \qquad 1 \leqq i \leqq n,$$

$$r_{n+1}[\alpha, u_0] \rightarrow 0,$$

where $\pi_H{}^{-1} e^{\alpha}{}_{i}$ is the reciprocal image of the section $e^{\alpha}{}_{i}$; consequently, $\mathcal{B}(\pi_H E) \cap \theta_{;2}{}^{-1}(U''_{\alpha,u_0})$ consists of all points $(\mathfrak{g}, \mathfrak{h})$ which satisfy

$$\omega^{\alpha}{}_{n+u_0}(\mathfrak{g}) = 0,$$

which clearly gives us that $\mathcal{B}(\pi_H E) \cap \theta_{;2}{}^{-1}(U''_{\alpha,u_0})$ is identical to

$$\mathfrak{N}_{\theta} \cap \theta_{;2}{}^{-1}(U''_{\alpha,u_0}).$$

Thus the subvariety $\mathfrak{N}_\theta$ on $\mathfrak{B}(\theta)$ carries the structure of a projective fiber bundle in two ways; the restriction of $\theta_{;2}$ to $\mathfrak{N}_\theta$ equips $\mathfrak{N}_\theta$ with the structure of $\mathfrak{B}(\pi_H E)$, and the restriction of $\theta_{;1}$ to $\mathfrak{N}_\theta$ equips $\mathfrak{N}_\theta$ with the structure of $\mathfrak{B}(\pi_E H)$— since $N(\mathfrak{B}(G), \mathfrak{B}(E)) = \pi_E H \otimes B^{-1}(E)$.

It is a straightforward matter to check that

$$B(R(\theta)) = \theta_{;1} B(G);$$

the restriction of $\theta_{;1}B(G)$ to $\mathfrak{N}_\theta$ is equal to the reciprocal image of $B(E)$ with respect to the mapping from $\mathfrak{N}_\theta$ onto $\mathfrak{B}(E)$ which is the restriction of $\theta_{;1}$ to $\mathfrak{N}_\theta$.

§ 18. Let V be a non-singular subvariety of dimension $r-1$ on X (dim. $X = r$). E and G are locally free sheaves defined on X as in § 16; but here we assume that ψ is an isomorphism from G into E, and that the residue class sheaf Q of E modulo the image of G by ψ is the extension to X of a locally free sheaf defined on V which we continue to denote as Q. We suppose that the covering $\{U_\alpha\}_{\alpha \in J}$ of X has the property that for each α, there exists a regular function x^α on U_α which generates the ideal determined by V in the local ring of X at each point on U_α, and that the restriction of Q to $V \cap U_\alpha$ (assuming that $V \cap U_\alpha$ is non-empty) is a free sheaf of dimension q defined on $V \cap U_\alpha$ generated by sections

$$l^\alpha{}_1, \cdots, l^\alpha{}_q$$

of Q over $V \cap U_\alpha$. It is permissible to suppose that the sections $e^\alpha{}_1, \cdots, e^\alpha{}_n$ are such that the restriction to U_α of the homomorphism from E onto Q is described by

$$e^\alpha{}_i \to l^\alpha{}_i, \qquad 1 \leqq i \leqq q,$$
$$e^\alpha{}_{q+u} \to 0, \qquad 1 \leqq u \leqq n-q,$$

and the restriction of the kernel of this homomorphism to U_α is generated by

$$x^\alpha e^\alpha{}_i \to 0, \qquad 1 \leqq i \leqq q,$$
$$e^\alpha{}_{q+u} \to 0, \qquad 1 \leqq u \leqq n-q;$$

furthermore, we can suppose that the sections $g^\alpha{}_1, \cdots, g^\alpha{}_n$ (here we must have $m = n$) are such that the restriction of ψ to U_α is described by

$$g^\alpha{}_i \to x^\alpha e^\alpha{}_i, \qquad 1 \leqq i \leqq q,$$
$$g^\alpha{}_{q+j} \to e^\alpha{}_{q+u}, \qquad 1 \leqq u \leqq n-q.$$

Thus we have the exact sequence

$$0 \to G \to E \to Q \to 0,$$

where Q is the extension to X of a locally free sheaf defined on V.

We suppose that x^α is the constant function 1 for each α such that $V \cap U_\alpha$ is empty. For each ordered pair, we introduce the function $D^{\beta\alpha}$ according to

$$x^\alpha D^{\beta\alpha} = x^\beta;$$

$D^{\beta\alpha}$ is a regular function on $U_\alpha \cap U_\beta$ which vanishes at no point of $U_\alpha \cap U_\beta$, and we have

$$D^{\alpha\gamma} D^{\alpha\beta} D^{\beta\alpha} = 1,$$
$$D^{\alpha\beta} D^{\beta\alpha} = 1.$$

We have the transition laws

$$e^\alpha{}_i = \sum_{h=1}^{n} E^\beta{}_h{}^\alpha{}_i e^\beta{}_h, \qquad 1 \leqq i \leqq n,$$

$$g^\alpha{}_i = \sum_{h=1}^{n} G^\beta{}_h{}^\alpha{}_i g^\beta{}_h, \qquad 1 \leqq i \leqq n,$$

for E and G respectively; but in view of the homomorphism ψ, we must have

$$G^\beta{}_h{}^\alpha{}_i = (D^{\beta\alpha})^{-1} E^\beta{}_h{}^\alpha{}_i, \qquad 1 \leqq i \leqq q, 1 \leqq h \leqq q,$$
$$G^\beta{}_{q+n}{}^\alpha{}_i = (D^{\beta\alpha})^{-1} x^\beta E^\beta{}_{q+u}{}^\alpha{}_i, \qquad 1 \leqq i \leqq q, 1 \leqq u \leqq n-q,$$
$$x^\beta G^\beta{}_h{}^\alpha{}_{q+v} = E^\beta{}_h{}^\alpha{}_{q+v}, \qquad 1 \leqq h \leqq q, 1 \leqq v \leqq n-q,$$
$$G^\beta{}_{q+n}{}^\alpha{}_{q+v} = E^\beta{}_{q+u}{}^\alpha{}_{q+v}, \qquad 1 \leqq u, v \leqq n-q.$$

This gives the transition laws

$$g^\alpha{}_i = (D^{\beta\alpha})^{-1} \Big(\sum_{h=1}^{q} E^\beta{}_h{}^\alpha{}_i g^\beta{}_h + x^\beta \sum_{u=1}^{n-q} E^\beta{}_{q+u}{}^\alpha{}_i g^\beta{}_{q+u} \Big)$$

for $1 \leqq i \leqq q$, and

$$g^\alpha{}_{q+v} = (x^\beta)^{-1} \sum_{h=1}^{q} E^\beta{}_h{}^\alpha{}_{q+v} g^\beta{}_h + \sum_{u=1}^{n-q} E^\beta{}_{q+u}{}^\alpha{}_{q+v} g^\beta{}_{q+v}$$

for $1 \leqq v \leqq n-q$; in particular, we observe that $(x^\beta)^{-1} E^\beta{}_h{}^\alpha{}_{q+v}$ is a regular function on $U_\alpha \cap U_\beta$ for all $1 \leqq h \leqq q$, $1 \leqq v \leqq n-q$.

Let E', resp. G', denote the induced sheaf of E, resp. G, on the subvariety V; let ψ' denote the induced homomorphism of ψ. E' and G' are locally free sheaves of dimension n defined on V; the induced sheaf of Q on V is clearly the sheaf Q. We have the exact sequence

$$G' \xrightarrow{\psi'} E' \to Q \to 0$$

of locally free sheaves defined on V; but $\operatorname{Ker}[\psi']$ is not the sheaf zero. In fact, the restriction of G' to $V \cap U_\alpha$ (assuming that this is not empty) is a free sheaf of dimension n defined on $V \cap U_\alpha$ and it is generated by the sections $g'^\alpha{}_1, \cdots, g'^\alpha{}_n$ which are the induced sections of $g^\alpha{}_1, \cdots, g^\alpha{}_n$ respectively on $V \cap U_\alpha$. The restriction of $\operatorname{Ker}[\psi']$ to $V \cap U_\alpha$ is a free sheaf of dimension q generated by the sections

$$g'^\alpha{}_1, \cdots, g'^\alpha{}_q;$$

we have the transition laws

$$g'^\alpha{}_i = (D'^{\beta\alpha})^{-1} \sum_{h=1}^{q} E'^\beta{}_h{}^\alpha{}_i g'^\beta{}_h, \qquad 1 \leqq h \leqq q,$$

where $D'^{\beta\alpha}$, resp. $E'^\beta{}_h{}^\alpha{}_i$, is the induced function of $D^{\beta\alpha}$, resp. $E^\beta{}_h{}^\alpha{}_i$, on $V \cap U_\alpha \cap U_\beta$. On the other hand, we have for the sheaf Q the transition laws

$$l^\alpha{}_i = \sum_{h=1}^{q} E'^\beta{}_h{}^\alpha{}_i l^\beta{}_h, \qquad 1 \leqq h \leqq q;$$

this proves that $\operatorname{Ker}[\psi']$ is isomorphic to the product sheaf of Q with a locally free sheaf of dimension one defined on V.

$\mathcal{L}(-V)$ is the sheaf of germs of rational function on X which are multiples of the divisor $-V$ on X. The restriction of $\mathcal{L}(-V)$ to U_α is a free sheaf of dimension one and the function $(x^\alpha)^{-1}$ is a generating section. We have the transition law

$$(x^\alpha)^{-1} = D^{\beta\alpha}(x^\beta)^{-1}$$

on $U_\alpha \cap U_\beta$. It is evident that the basic divisor class $\Theta(\mathcal{L}(-V))$ of $\mathcal{L}(-V)$ is the divisor class of V. Let $(\mathcal{L}(-V))'$ denote the induced sheaf of $\mathcal{L}(-V)$ on the subvariety V. Then we obtain by inspection that

$$\operatorname{Ker}[\psi'] \otimes (\mathcal{L}(-V))' = Q,$$

or better,

$$\operatorname{Ker}[\psi'] = Q \otimes (\mathcal{L}(V))',$$

where $(\mathcal{L}(V))'$ is the induced sheaf on V of the sheaf $\mathcal{L}(V)$ of germs of rational functions on X which are multiples of the divisor V. Thus we have obtained the exact sequence

$$0 \to Q \otimes (\mathcal{L}(V))' \to G' \xrightarrow{\psi'} E' \to Q \to 0$$

of locally free sheaves defined on V; this is called the "subordinate exact sequence" of the exact sequence

$$0 \to G \xrightarrow{\psi} E \to Q \to 0.$$

The sheaf $\mathrm{Im}[\psi']$ is a locally free sheaf of dimension $n-q$ defined on V; we denote this sheaf by $M(\psi)$. $M(\psi)$ is a subsheaf of E', and the restriction of $M(\psi)$ to $V \cap U_\alpha$ is a free sheaf generated by the sections

$$e'^{\alpha}_{q+1}, \cdots, e'^{\alpha}_{n}.$$

With reference to $M(\psi)$, we have the transition laws

$$e'^{\alpha}_{q+v} = \sum_{u=1}^{n-q} E'^{\beta}{}_{q+u}{}^{\alpha}{}_{q+v} e'_{q+u}, \qquad 1 \leqq v \leqq n-q,$$

since $E^{\beta}{}_{h}{}^{\alpha}{}_{q+v} = x^{\beta} G^{\beta}{}_{h}{}^{\alpha}{}_{q+v}$ for $1 \leqq h \leqq q$, $1 \leqq v \leqq n-q$, which forces $E'^{\beta}{}_{h}{}^{\alpha}{}_{q+v}$ to be the function zero on V. Thus the subordinate exact sequence is composed of the exact sequences

$$0 \to Q \otimes (\mathscr{L}(V))' \to G' \to M(\psi) \to 0,$$
$$0 \to M(\psi) \to E' \to Q \to 0$$

of locally free sheaves defined on V.

§ 19. Consider the exact sequence

$$0 \to G \to E \to Q \to 0$$

of § 18. We shall prove that the graph $\mathscr{C}(\psi)$ of $\mathscr{B}(\psi)$ is a non-singular variety.

We form the variety $Y(\psi)$ and the projection Σ_ψ from $Y(\psi)$ onto X, as in § 16. $\Sigma_\psi^{-1}(U_\alpha)$ is the product variety $U_\alpha \times P_\alpha \times P'_\alpha$. A point $(\mathfrak{e}, \mathfrak{g})$ on $\mathscr{C}(\psi) \cap \Sigma_\psi^{-1}(U_\alpha)$ must satisfy

$$\rho_1 \omega^{\alpha}{}_i(\mathfrak{g}) = \rho_2 x^{\alpha}(\mathfrak{p}) \tau^{\alpha}{}_i(\mathfrak{e}), \qquad 1 \leqq i \leqq q,$$
$$\rho_1 \omega^{\alpha}{}_{q+u}(\mathfrak{g}) = \rho_2 \tau^{\alpha}{}_{q+u}(\mathfrak{e}), \qquad 1 \leqq u \leqq n-q,$$

where $\mathfrak{p} = \Sigma_\psi(\mathfrak{e}, \mathfrak{g}) = \pi_E(\mathfrak{e}) = \pi_G(\mathfrak{g})$. The case where $\mathfrak{p}$ is not a point V is essentially trivial; in fact, we then have that $\mathscr{B}(\psi)$ bi-regularly maps some open neighborhood of $\mathfrak{e}$ on $\mathscr{B}(E)$ onto an open neighborhood of $\mathfrak{g}$ on $\mathscr{B}(G)$. Assume that $\mathfrak{p}$ lies on V so that $x^{\alpha}(\mathfrak{p}) = 0$.

Case I: $\tau^{\alpha}{}_{q+u_0}(\mathfrak{e}) \neq 0$, $1 \leqq u_0 \leqq n-q$. Then we have

$$\omega^{\alpha}{}_i(\mathfrak{g}) - x^{\alpha}(\mathfrak{p}) (\tau^{\alpha}{}_i / \tau^{\alpha}{}_{q+u_0})(\mathfrak{e}) \omega^{\alpha}{}_{q+u_0}(\mathfrak{g}) = 0, \qquad 1 \leqq i \leqq q,$$
$$\omega^{\alpha}{}_{q+u}(\mathfrak{g}) - (\tau^{\alpha}{}_{q+u} / \tau^{\alpha}{}_{q+u_0})(\mathfrak{e}) \omega^{\alpha}{}_{q+u_0}(\mathfrak{g}) = 0, \qquad 1 \leqq u \leqq n-q.$$

This forces $\omega^{\alpha}{}_{q+u_0}(\mathfrak{g}) \neq 0$. Consequently, the $n-1$ functions

$$\omega^{\alpha}{}_i / \omega^{\alpha}{}_{q+u_0} - x^{\alpha}(\tau^{\alpha}{}_i / \tau^{\alpha}{}_{q+u_0}), \qquad 1 \leqq i \leqq q,$$
$$\omega^{\alpha}{}_{q+u} / \omega^{\alpha}{}_{q+u_0} - \tau^{\alpha}{}_{q+u} / \tau^{\alpha}{}_{q+u_0}, \qquad 1 \leqq u \leqq n-q, u \neq u_0,$$

generate the ideal determined by $\mathcal{B}(\psi)$ in the local ring of $Y(\psi)$ at $(\mathfrak{e}, \mathfrak{g})$. It is clear that $(\mathfrak{e}, \mathfrak{g})$ is a simple point on $\mathcal{B}(\psi)$, and that the projection $\psi_{;1}$ from $\mathcal{B}(\psi)$ onto $\mathcal{B}(E)$ bi-regularly maps some open neighborhood of $(\mathfrak{e}, \mathfrak{g})$ on $\mathcal{B}(\psi)$ onto an open neighborhood of $\mathfrak{e}$ on $\mathcal{B}(E)$.

Case II: $\tau^{\alpha}{}_{q+u}(\mathfrak{e}) = 0$ for all $1 \leqq u \leqq n-q$; hence $\tau^{\alpha}{}_{i_0}(\mathfrak{e}) \neq 0$ for some i_0, $1 \leqq i_0 \leqq q$. Then we have

$$\omega^{\alpha}{}_{i}(\mathfrak{g}) - (\tau^{\alpha}{}_{i}/\tau^{\alpha}{}_{i_0})(\mathfrak{e})\omega^{\alpha}{}_{i_0}(\mathfrak{g}) = 0, \qquad 1 \leqq i \leqq n-q,$$

$$x^{\alpha}(\mathfrak{p})\omega^{\alpha}{}_{q+u}(\mathfrak{g}) - (\tau^{\alpha}{}_{q+u}/\tau^{\alpha}{}_{i_0})(\mathfrak{e})\omega^{\alpha}{}_{i_0}(\mathfrak{g}) = 0, \qquad 1 \leqq u \leqq q,$$

$$(\tau^{\alpha}{}_{q+u_1}/\tau^{\alpha}{}_{i_0})(\mathfrak{e})\omega^{\alpha}{}_{q+u_2}(\mathfrak{g}) - (\tau^{\alpha}{}_{q+u_2}/\tau^{\alpha}{}_{i_0})(\mathfrak{e})\omega^{\alpha}{}_{q+u_1}(\mathfrak{g}) = 0,$$
$$1 \leqq u_1, u_2 \leqq n-q.$$

This forces at least one of the following to be not equal to zero:

$$\omega^{\alpha}{}_{i_0}(\mathfrak{g}), \omega^{\alpha}{}_{q+1}(\mathfrak{g}), \cdots, \omega^{\alpha}{}_{n}(\mathfrak{g}).$$

Assume first that $\omega^{\alpha}{}_{i_0}(\mathfrak{g}) \neq 0$; then the $n-1$ functions

$$\omega^{\alpha}{}_{i}/\omega^{\alpha}{}_{i_0} - \tau^{\alpha}{}_{i}/\tau^{\alpha}{}_{i_0}, \qquad 1 \leqq i \leqq q, i \neq i_0,$$

$$\tau^{\alpha}{}_{q+u}/\tau^{\alpha}{}_{q+u_0} - x^{\alpha}(\omega^{\alpha}{}_{q+u}/\omega^{\alpha}{}_{i_0}), \qquad 1 \leqq u \leqq n-q,$$

generate the ideal determined by $\mathcal{B}(\psi)$ in the local ring of $Y(\psi)$ at $(\mathfrak{e}, \mathfrak{g})$. Next assume that $\omega^{\alpha}{}_{q+u_0}(\mathfrak{g}) \neq 0$ for some u_0, $1 \leqq u_0 \leqq n-q$; then the $n-1$ functions

$$\omega^{\alpha}{}_{i}/\omega^{\alpha}{}_{q+u_0} - (\tau^{\alpha}{}_{i}/\tau^{\alpha}{}_{i_0})\omega^{\alpha}{}_{i_0}/\omega^{\alpha}{}_{q+u_0}, \qquad 1 \leqq i \leqq q, i \neq i_0,$$

$$\tau^{\alpha}{}_{q+u}/\tau^{\alpha}{}_{1_0} - (\tau^{\alpha}{}_{q+u_0}/\tau^{\alpha}{}_{i_0})\omega^{\alpha}{}_{q+u}/\omega^{\alpha}{}_{q+u_0}, \qquad 1 \leqq u \leqq n-q, u \neq u_0,$$

$$x^{\alpha} - (\tau^{\alpha}{}_{q+u_0}/\tau_{i_0})\omega^{\alpha}{}_{i_0}/\omega^{\alpha}{}_{q+u_0}$$

generate the ideal determined by $\mathcal{B}(\psi)$ in the local ring of $Y(\psi)$ at $(\mathfrak{e}, \mathfrak{g})$. In both subcases, it follows immediately that $(\mathfrak{e}, \mathfrak{g})$ is a simple point on $\mathcal{B}(\psi)$. We have by inspection that the open set $\psi_{;1}{}^{-1}(U_{\alpha, i_0})$ is obtained by monoidal transformation of U_{α, i_0} centered on the non-singular subvariety which determines the ideal generated by the $n-q+1$ functions

$$x^{\alpha}$$

$$\tau^{\alpha}{}_{q+u}/\tau^{\alpha}{}_{i_0}, \qquad 1 \leqq u \leqq n-q.$$

in the local ring of each point on U_{α, i_0}.

We have proved that $\mathcal{B}(\psi)$ is a non-singular variety, and that the projection $\psi_{;1}$ is an anti-monoidal transformation whose center is a non-singular

subvariety of dimension $r+q-2=(r-1)+(q-1)$ on $\mathfrak{B}(E)$. Let us denote this subvariety, temporarily, by W. We shall exhibit a specific biregular mapping of the dual projective bundle $\mathfrak{B}(Q)$ of Q onto W. First let us observe that a point $\mathfrak{e}$ on $U_\alpha \times P_\alpha$ belongs to $W \cap (U_\alpha \times P_\alpha)$ if and only if:

(a) $\mathfrak{p}=\pi_E(\mathfrak{e})$ is a point on V, so that $x^\alpha(\mathfrak{p})=0$.

(b) $\tau^\alpha{}_{q+u}(\mathfrak{e})=0, \qquad 1 \leqq u \leqq n-q;$

next if $\mathfrak{e}$ lies on $W \cap (U_\alpha \times P_\alpha) \cap (U_\beta \times P_\beta)$, then we must have

$$\tau^\alpha{}_i(\mathfrak{e})=\rho \sum_{h=1}^{q} E^\beta{}_h{}^\alpha{}_i(\mathfrak{p})\tau^\beta{}_h(\mathfrak{e}), \qquad 1 \leqq i \leqq q,$$

since $\tau^\beta{}_{q+u}(\mathfrak{e})=0$ for all $1 \leqq u \leqq n-q$. This proves that the restriction of π_E to W equips W with the structure of a projective fiber bundle whose base space is V. With reference to the sheaf Q, we had the transition laws

$$l^\alpha{}_i=\sum_{h=1}^{q} E'^\beta{}_h{}^\alpha{}_i l^\beta{}_h, \qquad 1 \leqq i \leqq q,$$

where $E'^\beta{}_h{}^\alpha{}_i$ is the induced function of $E^\beta{}_h{}^\alpha{}_i$ on the subvariety V; it follows by inspection that we can identify the bundle $\mathfrak{B}(Q)$ with W according to the rule that the induced homogeneous coordinates $\tau^\alpha{}_1, \cdots, \tau^\alpha{}_q$ on $W \cap (U_\alpha \times P_\alpha)$ are paired with the sections $l^\alpha{}_1, \cdots, l^\alpha{}_q$ over U_α with reference to $\mathfrak{B}(Q)$. The restriction of π_E to $\mathfrak{B}(Q)$ is equal to π_Q; the induced sheaf of $B(E)$ on the subvariety $\mathfrak{B}(Q)$ is equal to $B(Q)$.

Let $\mathfrak{N}_\psi$ denote the anti-center of $\psi_{;1}$; $\mathfrak{N}_\psi$ is a non-singular subvariety of dimension $r+n-2$ on $\mathcal{B}(\psi)$. From § 16, there is the locally free sheaf $S(\psi)$ of dimension one defined on $\mathcal{B}(\psi)$ and with the property

$$\psi_{;2}B(G) \otimes S(\psi)=\psi_{;1}B(E);$$

in the present situation, we obtain by an easy calculation that

$$S(\psi)=\mathcal{L}(-\mathfrak{N}_\psi),$$

recalling that $\mathcal{L}(-\mathfrak{N}_\psi)$ is the sheaf of germs of rational functions on $\mathcal{B}(\psi)$ which are multiples of the divisor $-\mathfrak{N}_\psi$. Thus the basic class $\Theta(S(\psi))$ is the divisor class of $\mathfrak{N}_\psi$, and we have the important formula

$$\psi_{;2}{}^*(\Theta(G))+\Theta(S(\psi))=\psi_{;1}{}^*(\Theta(E));$$

$\psi_{;1}{}^*(\Theta(E))$, resp. $\psi_{;2}{}^*(\Theta(G))$, is the reciprocal image of the divisor class $\Theta(E)$, resp. $\Theta(G)$, with respect to $\psi_{;1}$, resp. $\psi_{;2}$.

Let $U_{\alpha;i,p}$ $(\alpha \in J,\ 1 \leqq i, p \leqq n)$ denote the open set on $\mathcal{B}(\psi)$ consisting of all points $(\mathfrak{e}, \mathfrak{g})$ on $\mathcal{B}(\psi) \cap (\Sigma_\psi)^{-1}(U_\alpha)$ which satisfy $\tau^\alpha{}_i(\mathfrak{e}) \neq 0$, $\omega^\alpha{}_p(\mathfrak{g}) \neq 0$. If $U_\alpha \cap V$ is empty, then $\mathfrak{N}_\psi \cap U_{\alpha;i,p}$ is empty. Assume that $U_\alpha \cap V$ is non-empty. If $i > q$, then $\mathfrak{N}_\psi \cap U_{\alpha;i,p}$ is empty. If $i \leqq q$ and $p > q$, then the function

$$\tau^\alpha{}_p/\tau^\alpha{}_i$$

generates the ideal determined by $\mathfrak{N}_\psi$ in the local ring of $\mathcal{B}(\psi)$ at any point on $U_{\alpha;i,p}$. If $i \leqq q$ and $p \leqq q$, then the function

$$x^\alpha$$

generates the ideal determined by $\mathfrak{N}_\psi$ in the local ring of $\mathcal{B}(\psi)$ at any point on $U_{\alpha;i,p}$.

The projection $\psi_{;2}$ is anti-monoidal transformation from $\mathcal{B}(\psi)$ onto $\mathcal{B}(G)$ with a non-singular subvariety on $\mathcal{B}(G)$ for center. In fact, for a point $(\mathfrak{e}, \mathfrak{g})$ on $\mathcal{B}(\psi) \cap (\Sigma_\psi)^{-1}(U_\alpha)$, we must have

$$\rho_1 x^\alpha(\mathfrak{p})\omega^\alpha{}_{q+u}(\mathfrak{g}) = \rho_2 \tau^\alpha{}_{q+u}(\mathfrak{e}) \qquad 1 \leqq u \leqq n-q;$$

$$\rho_1 \omega^\alpha{}_i(\mathfrak{g}) = \rho_2 \tau^\alpha{}_i(\mathfrak{e}), \qquad 1 \leqq i \leqq q,$$

the discussion in cases I and II is simply repeated but now with $\omega^\alpha{}_1, \cdots, \omega^\alpha{}_q$ playing the former roles of $\tau^\alpha{}_{q+1}, \cdots, \tau^\alpha{}_n$. The center of $\psi_{;2}$ is a subvariety Z of dimension $r + n - q - 2 = (r-1) + (n-q-1)$ on $\mathcal{B}(G)$; the $q+1$ functions

$$\omega^\alpha{}_i/\omega^\alpha{}_{q+u}, \qquad 1 \leqq i \leqq q,$$

$$x^\alpha$$

generate the ideal determined by Z in the local ring of $\mathcal{B}(G)$ at each point of $U'_{\alpha,q+u}$; $Z \cap U'_{\alpha,i}$ is empty if $1 \leqq i \leqq q$. The restriction of π_G to Z equips Z with the structure of a projective fiber bundle whose base space is V; as such Z is the dual projective bundle of the locally free sheaf $M(\psi)$ defined on V, and we set $Z = \mathcal{B}(M(\psi))$. $\pi_G{}^{-1}(V)$, the portion of $\mathcal{B}(G)$ over V, is a non-singular subvariety of dimension $r + n - 2$ on $\mathcal{B}(G)$; the restriction of π_G to that subvariety equips it with the structure of the dual projective bundle $\mathcal{B}(G')$ of the induced sheaf G' of G on the subvariety V. We observe that $\mathcal{B}(M(\psi))$ is a subvariety on $\mathcal{B}(G')$.

Let $\mathfrak{M}_\psi$ denote the anti-center of $\psi_{;2}$. If $U_\alpha \cap V$ is empty, then $\mathfrak{M}_\psi \cap U_{\alpha;i,p}$ is empty. Assume that $U_\alpha \cap V$ is not empty. If $p \leqq q$, then $\mathfrak{M}_\psi \cap U_{\alpha;i,p}$ is empty. If $p > q$ and $i \leqq q$, then the function

$$\omega^\alpha{}_i/\omega^\alpha{}_p$$

generates the ideal determined by $\mathfrak{M}_\psi$ in the local ring of $\mathcal{B}(\psi)$ at any point on $U_{\alpha;i,p}$; if $p > q$ and $i > q$, then the function

$$x^\alpha$$

generates the ideal determined by $\mathfrak{M}_\psi$ in the local ring of $\mathcal{B}(\psi)$ at any point on $U_{\alpha;i,p}$. The divisor $\psi_{;2}{}^*(\mathfrak{B}(G'))$, which is the reciprocal image of the divisor $\mathfrak{B}(G')$ on $\mathfrak{B}(G)$ with respect to $\psi_{;2}$ and also the total transform of $\mathfrak{B}(G')$ with respect to the inverse of $\psi_{;2}$, is the divisor $\mathfrak{M}_\psi + \mathfrak{N}_\psi$; that is to say,

$$\psi_{;2}{}^*(\mathfrak{B}(G')) = \mathfrak{M}_\psi + \mathfrak{N}_\psi.$$

The intersection cycle $\mathfrak{M}_\psi \circ \mathfrak{N}_\psi$ is a non-singular subvariety of dimension $r+n-2$ on $\mathcal{B}(\psi)$; the restriction of $\psi_{;2}$ to $\mathfrak{N}_\psi$ is an anti-monoidal transformation of $\mathfrak{N}_\psi$ onto $\mathfrak{B}(G')$ whose anti-center is $\mathfrak{M}_\psi \circ \mathfrak{N}_\psi$ and with center $\mathfrak{B}(M(\psi))$ as subvariety on $\mathfrak{B}(G')$. The restriction of $\psi_{;1}$ to $\mathfrak{M}_\psi$ is an anti-monoidal transformation from $\mathfrak{M}_\psi$ onto the non-singular subvariety $\mathfrak{B}(E')$ on $\mathfrak{B}(E)$, where $\mathfrak{B}(E')$ is the dual projective bundle of the induced sheaf E' of E on the subvariety V on X, whose anti-center is $\mathfrak{M}_\psi \circ \mathfrak{N}_\psi$ and with center $\mathfrak{B}(Q)$ as subvariety on $\mathfrak{B}(E')$. $\psi_{;1}{}^*(\mathfrak{B}(E'))$, the reciprocal image of the divisor $\mathfrak{B}(E')$ on $\mathfrak{B}(E)$, is the divisor $\mathfrak{M}_\psi + \mathfrak{N}_\psi$; consequently, we have

$$\psi_{;2}{}^*(\mathfrak{B}(G')) = \mathfrak{M}_\psi + \mathfrak{N}_\psi = \psi_{;1}{}^*(\mathfrak{B}(E')).$$

§20. We shall prove that the dual projective bundle $\mathfrak{B}(E)$ admits a projective model. Let $\mathcal{O}_X{}^N$ denote the direct sum $\mathcal{O}_X + \cdots + \mathcal{O}_X$ taken N times. It is obvious that $\mathfrak{B}(\mathcal{O}_X{}^N)$ is the product variety of X and a projective space of dimension $N-1$; hence, $\mathfrak{B}(\mathcal{O}_X{}^N)$ admits a projective model since we have supposed that X admits a projective model. With respect to a specific projective imbedding of X we denote by D_h the locally free sheaf of dimension one defined on X such that $\Theta(D_h)$ is the divisor class of the linear system of hypersurface sections of degree h on X. Now we have

$$\mathfrak{B}(\mathcal{O}_X{}^N \otimes D_h) = \mathfrak{B}(\mathcal{O}_X{}^N),$$

and it is an elementary fact (the construction of the projective model of the product of two projective models) that for any $h > 0$, the linear system of positive divisors from the divisor class $\Theta(\mathcal{O}_X{}^N \otimes D_h)$ serves as a system of hyperplane sections for a projective model for $\mathfrak{B}(\mathcal{O}_X{}^N)$.

On the other hand, we have, from one of Serre's fundamental theorems, that if we are given a sheaf E, then, for all sufficiently large positive integers

h_0, we can choose $\mathcal{O}_X{}^N$ and a homomorphism ψ of $\mathcal{O}_X{}^N$ onto $E \otimes D_{h_0}$. Assuming that E is a locally free sheaf and applying the results of § 17, we have that $\mathcal{B}(\psi)$ is a bi-regular mapping of $\mathcal{B}(E)$—which is the same as $\mathcal{B}(E \otimes D_{h_0})$—onto a subvariety on $\mathcal{B}(\mathcal{O}_X{}^N)$. This proves that $\mathcal{B}(E)$ admits a projective model and that the linear system of positive divisors from $\Theta(E \otimes D_{h_0+1})$ serves a system of hyperplane sections for a projective model of $\mathcal{B}(E)$.

IV. The Unicity of the $\mathcal{A}$-genus.

§ 21. By an arithmetic functional, we shall mean a mapping $\mathcal{A}$ which assigns a rational number $\mathcal{A}(X)$ to every non-singular projective model X and satisfies the following axioms:

Axiom I: (Normalization) $\mathcal{A}(P) = 1$ for any projective space; $\mathcal{A}(\emptyset) = 0$, where $\emptyset$ is the empty variety.

Axiom II: (Modular Law) X is a non-singular projective model of dimension r; A, A' are non-singular subvarieties of dimension $r-1$ on X; the intersection cycle $A \circ A'$ is proper and

$$A \circ A' = \sum_{i=1}^{s} V_i,$$

where V_i is a non-singular subvariety of dimension $r-2$ on X for all $1 \leqq i \leqq s$, and $V_i \cap V_j$ is empty if $i \neq j$, $1 \leqq i, j \leqq s$; finally, there is a non-singular subvariety A_1 of dimension $r-1$ on X such that the divisor A_1 is linearly equivalent to the divisor $A + A'$. In these circumstances, we require that

$$\mathcal{A}(A_1) = \mathcal{A}(A) + \mathcal{A}(A') - \sum_{i=1}^{s} \mathcal{A}(V_i).$$

Axiom III: (Fiber Law) Y, X are non-singular projective models; $\Phi: Y \rightarrow X$ is a rational transformation from Y onto X which satisfies either of the following conditions:

(a) Φ equips Y with the structure of the dual projective bundle of some locally free sheaf defined on X;

(b) Y is obtained by monoidal transformation of X centered on a non-singular subvariety on X with Φ the anti-monoidal transformation. Then we require that

$$\mathcal{A}(Y) = \mathcal{A}(X).$$

In this chapter, we assume the existence of an arithmetic functional $\mathcal{A}$; $\mathcal{A}(X)$ is called the $\mathcal{A}$-genus of the variety X. The main issue will be to prove that $\mathcal{A}$ is unique.

§ 22. Our aim is to define the "virtual" $\mathcal{A}$-genus of an arbitrary divisor class on a non-singular projective model X. To this end, we must repeat for the $\mathcal{A}$-genus the familiar arguments used to define the virtual arithmetic genus.

We note first that $\mathcal{A}(X)$ is a bi-regular invariant of the non-singular projective model X; this is the weak form of the Fiber Law (Axiom III) which corresponds to the case where Φ is a bi-regular mapping.

Next, if A, A_1 are non-singular subvarieties of dimension $r-1$ on X ($\dim. X = r$) and if A, A_1 are linearly eqivalent as divisors on X, then, as follows from the Modular Law (Axiom II) by taking A' equal to the empty variety and using Axiom I,

$$\mathcal{A}(A_1) = \mathcal{A}(A). \tag{1}$$

Let x be a divisor class on X, and assume that there exists a non-singular subvariety A of dimension $r-1$ which, as divisor, belongs to x. In this situation, we define $\mathcal{A}(x)$, the $\mathcal{A}$-genus of the divisor class x, according to

$$\mathcal{A}(x) = \mathcal{A}(A);$$

$\mathcal{A}(x)$ depends solely upon x, as follows from (1). (If $\dim. X = 1$, and if x contains a positive divisor A of type $\mathfrak{p}_1 + \cdots + \mathfrak{p}_s$ with $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ different points of X, then we define

$$\mathcal{A}(x) = \mathcal{A}(A) = s;$$

this definition is consistent with Axiom I and the Modular Law.)

Now assume that x is sufficiently ample (i.e., the positive divisors of x serve as a system of hyperplane sections for some projective imbedding of X). Then $\mathcal{A}(x)$ is defined as above. We define $\mathcal{A}(x^k)$ according to

$$\mathcal{A}(x^k) = \mathcal{A}(A_1 \circ \cdots \circ A_k),$$

where $A_1 \circ \cdots \circ A_k$ is the intersection cycle formed by k-general members of x; this cycle is a non-singular subvariety of co-dimension k on X (with the obvious modification if $k = f = \dim. X$); $\mathcal{A}(x^k)$ depends solely upon x, as follows by repeating the argument used to obtain (1); furthermore, if $k > r$, then $\mathcal{A}(x^k) = 0$, as follows from Axiom I.

Let W be a non-singular subvariety on X, and let $\{x\}_W$ denote the trace

of x on W. Then $\{x\}_W$ is a sufficiently ample divisor class on W; hence, $\mathcal{A}((\{x\}_W)^k)$ is defined for all k and we have

$$\mathcal{A}((\{x\}_W)^k) = \mathcal{A}(A_1 \circ \cdots \circ A_k \circ W).$$

Let $x, y, z, \cdots$ be sufficiently ample divisor classes on X. Then $x+y$, $x+z, y+z, \cdots$ are sufficiently ample divisor classes on X. We consider $\mathcal{A}((\{x\}_B)^k)$, where B is a sufficiently general member of y; it depends solely upon x and y and we set

$$\mathcal{A}((\{x\}_y)) = \mathcal{A}((\{x\}_B)).$$

Similarly, we set

$$\mathcal{A}((\{x\}_{y \circ z})^k) = \mathcal{A}((\{x\}_{B \circ C})^k),$$

where B, C are general members of y and z respectively, and we have

$$\mathcal{A}((\{x\}_{y \circ z})^k) = \mathcal{A}(A_1 \circ \cdots \circ A_k \circ B \circ C).$$

Let D be a non-singular member of the divisor class $y+z$. Then, as a consequence of the Modular Law, we have

$$\mathcal{A}((\{x\}_D)^k) = \mathcal{A}((\{x\}_B)^k) + \mathcal{A}((\{x\}_C)^k) - \mathcal{A}((\{x\}_{B \circ C})^k);$$

but we can rewrite this formula as

$$\mathcal{A}((\{x\}_{y+z})^k) = \mathcal{A}((\{x\}_y)^k) + \mathcal{A}((\{x\}_z)^k) - \mathcal{A}((\{x\}_{y \circ z})^k). \tag{2}$$

Define the quantity $\mathcal{K}(x)$ according to

$$\mathcal{K}(x) = \mathcal{A}(X) + \sum_{k=1}^{\infty} \mathcal{A}(x^k),$$

and observe that

$$\mathcal{K}(x) = \mathcal{A}(X) + \mathcal{K}(\{x\}_x). \tag{3}$$

The quantity $\mathcal{K}(\{x\}_W)$ is well defined for any non-singular subvariety W and

$$\mathcal{K}(\{x\}_W) = \mathcal{A}(W) + \sum_{k=1}^{\infty} \mathcal{A}(\{x\}_W{}^k).$$

In particular, $\mathcal{K}(\{x\}_B)$ is defined for any general member B of y, and we set

$$\mathcal{K}(\{x\}_y) = \mathcal{K}(\{x\}_B),$$

observing that $\mathcal{K}(\{x\}_y)$ depends solely upon x and y. Similarly, $\mathcal{K}(\{x\}_{y \circ z})$ is well defined and

$$\mathcal{K}(\{x\}_{y \circ z}) = \mathcal{K}(\{x\}_{B \circ C}).$$

From (2), we obtain

$$\mathcal{K}(\{x\}_{y+z}) = \mathcal{K}(\{x\}_y) + \mathcal{K}(\{x\}_z) - \mathcal{K}(\{x\}_{y \circ z}). \tag{4}$$

PROPOSITION 1.

(5) $$\mathcal{K}(x+y)=\mathcal{K}(x)+\mathcal{K}(\{x+y\}_y).$$

Proof. The proof is by induction on the dimension r of the ambient variety X. The proposition is evident if the ambient variety is of dimension one; we asume that it is true whenever the ambient variety is of dimension less than r. From (3), we have

$$\mathcal{K}(x+y)=\mathcal{A}(X)+\mathcal{K}(\{x+y\}_{x+y});$$

from (4), we have

$$\mathcal{K}(\{x+y\}_{x+y})=\mathcal{K}(\{x+y\}_x)+\mathcal{K}(\{x+y\}_y)-\mathcal{K}(\{x+y\}_{x\circ y});$$

the inductive assumption gives

$$\mathcal{K}(\{x\}_x=\mathcal{K}(\{x+y\}_x)-\mathcal{K}(\{x+y\}_{x\circ y}),$$

where the ambient variety is a general member of x; this gives

$$\mathcal{K}(\{x+y\}_{x+y})=\mathcal{K}(\{x\}_x)+\mathcal{K}(\{x+y\}_y),$$

and the proposition is proved by adding $\mathcal{A}(X)$ to both sides.

PROPOSITION 2.

(6) $$\mathcal{K}(\{y+z\}_{x+z})-\mathcal{K}(\{y+z\}_{y+z})=\mathcal{K}(\{y\}_x)-\mathcal{K}(\{y\}_y).$$

Proof. Applying (4), we have

$$\mathcal{K}(\{y+z\}_{x+z})=\mathcal{K}(\{y+z\}_x)+\mathcal{K}(\{y+z\}_z)-\mathcal{K}(\{y+z\}_{x\circ z});$$

from Proposition 1, we have

$$\mathcal{K}(\{y\}_x)=\mathcal{K}(\{y+z\}_x)-\mathcal{K}(\{y+z\}_{x\circ z}),$$

which proves that

(7) $$\mathcal{K}(\{y+z\}_{x+z})=\mathcal{K}(\{y\}_x)+\mathcal{K}(\{y+z\}_z);$$

similarly, we have

(8) $$\mathcal{K}(\{y+z\}_{y+z})=\mathcal{K}(\{y\}_y)+\mathcal{K}(\{y+z\}_z);$$

hence, (6) follows by substracting (8) from (7).

Let ϑ be an arbitrary divisor class on X. Then it is possible to choose sufficiently ample divisor classes x, y on X with the property $\vartheta=x-y$. We define $\mathcal{A}(\vartheta)$, the virtual $\mathcal{A}$-genus ϑ, according to

(9) $$\mathcal{A}(\vartheta)=\mathcal{A}(x-y)=\mathcal{K}(\{y\}_x)-\mathcal{K}(\{y\}_y).$$

On the basis of Proposition 2, it follows that $\mathcal{A}(\vartheta)$ depends solely upon ϑ. The explicit formula is

$$(10)\quad \mathcal{A}(\vartheta)=\mathcal{A}(A)+\sum_{k=1}^{r-1}\mathcal{A}(B_1\circ\cdots\circ B_k\circ A) \\ -\mathcal{A}(B)-\sum_{k=1}^{r-1}\mathcal{A}(B_1\circ\cdots\circ B_k\circ B),$$

where A (resp. $B, B_1,\cdots,B_{r-1}$) are general members of x (resp. y). If ϑ contains a non-singular member U, then

$$(11)\qquad \mathcal{A}(\vartheta)=\mathcal{A}(U);$$

for A, $B+U$ are linearly equivalent divisors on X and from the Modular Law, we have that

$$\mathcal{A}(A)=\mathcal{A}(B)+\mathcal{A}(U)-\mathcal{A}(B\circ U),$$

$$\mathcal{A}(B_1\circ\cdots\circ B_k\circ A)=\mathcal{A}(B_1\circ\cdots\circ B_k\circ B)+\mathcal{A}(B_1\circ\cdots\circ B_k\circ U) \\ -\mathcal{A}(B_1\circ\cdots\circ B_k\circ B\circ U).$$

(11) follows by summation of these last equations followed by subtraction of $\sum_{k=1}^{r-1}\mathcal{A}(B_1\circ\cdots\circ B_k\circ B)$.

For an arbitrary divisor class ϑ, we define $\mathcal{K}(\vartheta)$ according to

$$(12)\qquad \mathcal{K}(\vartheta)=\mathcal{K}(x-y)=\mathcal{K}(x)-\mathcal{K}(\{x\}_y),$$

where $\vartheta=x-y$, x,y sufficiently ample. (If ϑ is sufficiently ample, then this agrees with the previous definition as follows from Proposition 1.) From the equation $\mathcal{K}(x)=\mathcal{A}(X)+\mathcal{K}(\{x\}_x)$ and the definition of $\mathcal{A}(-\vartheta)$ $=K(y-x)$, it is evident that

$$(13)\qquad \mathcal{K}(\vartheta)=\mathcal{A}(X)-\mathcal{A}(-\vartheta).$$

THEOREM 1. *Let ϑ, ϑ_1 be arbitrary divisor classes on X with the property that $\vartheta-\vartheta_1$ contains a non-singular member T. Then*

$$(14)\qquad \mathcal{K}(\vartheta)=\mathcal{K}(\vartheta_1)+\mathcal{K}(\{\vartheta\}_T),$$

where $\{\vartheta\}_T$ denotes the trace of ϑ on T, and

$$(15)\qquad \mathcal{A}(\vartheta)=\mathcal{A}(\vartheta_1)+\mathcal{A}(T)-\mathcal{A}(\{\vartheta_1\}_T).$$

Proof. It is possible to choose sufficiently ample divisor classes x, y, y_1 such that $\vartheta=x-y$; $\vartheta_1=x-y_1$; thus T is a member of y_1-y. Now

$$\mathcal{K}(\vartheta_1)=\mathcal{K}(x-y_1)=\mathcal{K}(x)-\mathcal{K}(\{x\}_{y_1}),$$

and from the Modular Law,

$$\mathcal{K}(\{x\}_{y_1}) = \mathcal{K}(\{x\}_y) + \mathcal{K}(\{x\}_T) - \mathcal{K}(\{x\}_{y \circ T})$$

(where $\mathcal{K}(\{x\}_{y \circ T}) = \mathcal{K}(\{x\}_{B \circ T})$ with B a general member of y), which yields

$$\mathcal{K}(\vartheta_1) = \mathcal{K}(x) - \mathcal{K}(\{x\}_y) - \mathcal{K}(\{x\}_T) + \mathcal{K}(\{x\}_{y \circ T}). \tag{16}$$

But

$$\mathcal{K}(\vartheta) = \mathcal{K}(x) - \mathcal{K}(\{x\}_y)$$

and

$$\mathcal{K}(\{\vartheta\}_T) = \mathcal{K}(\{x\}_T) - \mathcal{K}(\{x\}_{y \circ T})$$

from the definition of $\mathcal{K}(\vartheta)$, $\mathcal{K}\{\vartheta\}_T)$; hence (14) follows from (16). We obtain immediately from (14) that

$$\mathcal{K}(-\vartheta_1) = \mathcal{K}(-\vartheta) + \mathcal{K}(-\{\vartheta_1\}_T). \tag{17}$$

But

$$\mathcal{K}(-\vartheta_1) = \mathcal{A}(X) - \mathcal{A}(\vartheta_1),$$
$$\mathcal{K}(-\vartheta) = \mathcal{A}(X) - \mathcal{A}(\vartheta),$$
$$\mathcal{K}(\{-\vartheta_1\}_T) = \mathcal{A}(T) - \mathcal{A}(\{\vartheta_1\}_T);$$

hence (15) follows from (17).

§ **23.** *X, Y* are non-singular projective models and Φ is a regular mapping from Y into X—eventually we shall assume that Φ satisfies one of the conditions required by the Fiber Law. Given a divisor class ϑ on X, then there is a divisor class $\Phi^*\vartheta$ on Y which is the reciprocal image of ϑ with respect to Φ. If D is a locally free sheaf of dimension one defined on X such that $\vartheta = \Theta(D)$, the basic class of D, then $\Phi^*\vartheta$ is the basic class of ΦD, the reciprocal image of D with respect to Φ.

Specifically, if there exists a posittive divisor A which belongs to ϑ and with the property that no component of A contains the image of Y in X—and this property will be automatically true if Φ satisfies one of the conditions of the Fiber Law—then there is the reciprocal image divisor Φ^*A on Y which is defined as follows: Let $f = 0$ be a local equation for A on an open subset U of X; then $\Phi^{-1}f = 0$ is a local equation for Φ^*A on $\Phi^{-1}(U)$, where $\Phi^{-1}f$ is the image of f under the natural homomorphism from $\Gamma(\mathcal{O}_U, U)$ into $\Gamma(\mathcal{O}_{\Phi^{-1}(U)}, \Phi^{-1}(U))$. The divisor Φ^*A belongs to the class $\Phi^*\vartheta$. Consequently, if x is a sufficiently ample divisor class on X, then for a general positive member A of x, we have that Φ^*A is an irreducible non-singular subvariety on X (with the usual modification for dim. $X = 1$).

Now assume that Φ satisfies the condition (a) in the Fiber Law, so that Φ equips Y with the structure of the dual projective bundle of some locally free sheaf S defined on X. Let x be a sufficiently ample divisor class on X and let A be a general member of X. Then the restriction of Φ to Φ^*A equips Φ^*A with the structure of the dual projective bundle of the induced sheaf of S on the subvariety A, and from the Fiber Law, we have $\mathfrak{a}(\Phi^*A) = \mathfrak{a}(A)$, which proves that

$$\mathfrak{a}(\Phi^*x) = \mathfrak{a}(x).$$

Let ϑ be an arbitrary divisor class on X and choose sufficiently ample divisor classes x, y with $\vartheta = x - y$; A, B are general members of x, y respectively. Applying Theorem 1 on the variety Y, we have

$$\mathfrak{a}(\Phi^*x) = \mathfrak{a}(\Phi^*\vartheta) + \mathfrak{a}(\Phi^*B) - \mathfrak{a}(\{\Phi^*\vartheta\}_{\Phi^*B})$$

since Φ^*B *is* a non-singular member of $\Phi^*\vartheta - \Phi^*x$, and on X, we have

$$\mathfrak{a}(x) = \mathfrak{a}(\vartheta) + \mathfrak{a}(B) - \mathfrak{a}(\{\vartheta\}_B).$$

Then we obtain, by induction on the dimension of the ambient variety X, that

$$\mathfrak{a}(\Phi^*\vartheta) = \mathfrak{a}(\vartheta); \tag{1}$$

for we have $\mathfrak{a}(x) = \mathfrak{a}(\Phi^*x)$, $\mathfrak{a}(B) = \mathfrak{a}(\Phi^*B)$ and, since $\dim. B$ is less than $\dim. X$, the inductive assumption gives $\mathfrak{a}(\{\vartheta\}_B) = \mathfrak{a}(\{\Phi^*\vartheta\}_{\Phi^*B})$. We have, as an immediate consequence of (1), that

$$\mathcal{K}(\Phi^*\vartheta) = \mathcal{K}(\vartheta). \tag{2}$$

Now assume that Φ satisfies the condition (b) of the Fiber Law, so that Y is obtained by monoidal transformation of X centered on a non-singular subvariety V on X and Φ is the anti-monoidal transformation. If A is a general member of a sufficiently ample divisor class x on X, then Φ^*A is a non-singular subvariety on Y and Φ^*A is the variety obtained by monoidal transformation of A centered on the intersection cycle $A \circ V$; the cycle $A \circ V$ is a non-singular subvariety of dimension $n-1$ on X if $\dim. V = n$, for a general member A of x; the restriction of Φ to Φ^*A is the anti-monoidal transformation. (If $n = 1$, then $A \circ V = \mathfrak{p}_1 + \cdots + \mathfrak{p}_s$ with $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ different points on A and Φ^*A is obtained by successive monoidal transformations of A centered at the points $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$.) The arguments of the previous paragraph can be repeated and we again have that

(1) $$\mathcal{A}(\Phi^*\vartheta) = \mathcal{A}(\vartheta),$$

(2) $$\mathcal{K}(\Phi^*\vartheta) = \mathcal{K}(\vartheta),$$

where Φ satisfies condition (b) of the Fiber Law.

THEOREM 2. *Let $\Phi\colon Y \to X$ satisfy one of the conditions of the Fiber Law. Then for any divisor class ϑ on X, we have*

$$\mathcal{A}(\Phi^*\vartheta) = \mathcal{A}(\vartheta),$$
$$\mathcal{K}(\Phi^*\vartheta) = \mathcal{K}(\vartheta).$$

The following proposition is essentially a corollary to Theorem 2, and it is used extensively in the next §.

PROPOSITION 3. *Let E be a locally free sheaf defined on X, and let $\Phi\colon Y \to X$ be as in the Fiber Law. Then*

(3) $$\mathcal{A}(\Theta(\Phi E)) = \mathcal{A}(\Theta(E)),$$

(4) $$\mathcal{K}(\Theta(\Phi E)) = \mathcal{K}(\Theta(E)).$$

Proof. We recall from § 8 of Chap. I that $\mathcal{B}(\Phi E)$ is the induced bundle of $\mathcal{B}(E)$ from the mapping $\Phi\colon Y \to X$ of the base spaces, $B(\Phi E)$ is the reciprocal image of $B(E)$ with respect to the fiber preserving map from $\mathcal{B}(\Phi E)$ to $\mathcal{B}(E)$ over $\Phi\colon Y \to X$, and $\Theta(\Phi E)$ is the reciprocal image of $\Theta(E)$ with respect to the fiber preserving map. But, by inspection, it is evident that the fiber preserving map from $\mathcal{B}(\Phi E)$ onto $\mathcal{B}(E)$ satisfies one of the conditions of the Fiber Law since Φ has this property, and our conclusions follow from Theorem 2.

§ 24. Given a locally free sheaf E defined on a non-singular projective model X, we define $\mathcal{K}(E)$, the $\mathcal{K}$-characteristic of the sheaf E, according to

(1) $$\mathcal{K}(E) = \mathcal{K}(\Theta(E)) = \mathcal{A}(\mathcal{B}(E)) - \mathcal{A}(-\Theta(E)).$$

For the sheaf $\mathcal{O}_X$ of local rings on X, we have

(2) $$\mathcal{K}(\mathcal{O}_X) = \mathcal{A}(X)$$

since $\Theta(\mathcal{O}_X)$ is the divisor class zero on X.

We have

(3) $$\mathcal{K}(B(E)) = \mathcal{K}(E)$$

since $\Theta(B(E)) = \Theta(E)$ and $\mathcal{A}(X) = \mathcal{A}(\mathcal{B}(E))$.

PROPOSITION 4. *X, Y are non-singular projective models, $\Phi\colon Y \to X$ satisfies one of the conditions of the Fiber Law. Then for any locally free sheaf E defined on X, we have*

$$\mathcal{K}(\Phi E) = \mathcal{K}(E). \tag{4}$$

Proof. This is a restatement of Proposition 3 which gave $\mathcal{K}(\Theta(\Phi E)) = \mathcal{K}(\Theta(E))$.

THEOREM 3. *Let*

$$0 \to H \xrightarrow{\theta} G \xrightarrow{\psi} E \to 0 \tag{5}$$

be an exact sequence of locally free sheaves defined on X. Then

$$\mathcal{K}(G) = \mathcal{K}(H) + \mathcal{K}(E).$$

Proof. We refer to the results and notations of § 17, Chap. III where the geometric resolution of (5) is discussed in detail. On the graph $\mathcal{G}(\theta)$, there are the projections $\theta_{;1}\colon \mathcal{G}(\theta) \to \mathcal{B}(G)$ and $\theta_{;2}\colon \mathcal{G}(\theta) \to \mathcal{B}(H)$; $\theta_{;1}$ is an anti-monoidal transformation whose center is the subvariety $\mathcal{B}(E)$ on $\mathcal{B}(G)$ and whose anti-center is $\mathcal{N}_\theta$; $\theta_{;2}$ equips $\mathcal{G}(\theta)$ with the structure of the dual projective bundle of the locally free sheaf $R(\theta)$ defined on $\mathcal{B}(H)$; thus both $\theta_{;1}$ and $\theta_{;2}$ satisfy one of the conditions of the Fiber Law. On $\mathcal{G}(\theta)$, we have

$$\theta_{;1}{}^*\Theta(G) = \theta_{;2}{}^*\Theta(H) + \Theta(S(\theta)) \tag{6}$$

and the non-singular subvariety $\mathcal{N}_\theta$ belongs to the divisor class $\Theta(S(\theta))$. Set $x = \theta_{;1}{}^*\Theta(G)$, $y = \theta_{;2}{}^*\Theta(H)$ and $s = \Theta(S(\theta))$ so that (6) is identical to

$$x = y + s. \tag{6'}$$

We can apply Theorem 1 since $s = x - y$ contains the non-singular member $\mathcal{N}_\theta$ and we have

$$\mathcal{K}(x) = \mathcal{K}(y) + \mathcal{K}(\{x\}_{\mathcal{N}_\theta}); \tag{7}$$

from Theorem 2, we have $\mathcal{K}(x) = \mathcal{K}(\Theta(G))$ and $\mathcal{K}(y) = \mathcal{K}(\Theta(H))$ since both $\theta_{;1}$ and $\theta_{;2}$ satisfy one of the conditions of the Fiber Law; $\{x\}_{\mathcal{N}_\theta}$, the trace of x on the subvariety $\mathcal{N}_\theta$, is equal to the reciprocal image of $\Theta(E)$ with respect to the restriction of $\theta_{;1}$ to $\mathcal{N}_\theta$ and, since this restriction mapping equips $\mathcal{N}_\theta$ with the structure of the dual projective bundle of the sheaf

$N(\mathscr{B}(G),\mathscr{B}(E))$, we can again apply Theorem 2 to obtain that $\mathscr{K}(\{x\}_{\mathscr{N}_0})=\mathscr{K}(\Theta(E))$. Substituting in (7), we obtain

$$\mathscr{K}(\Theta(G))=\mathscr{K}(\Theta(H))+\mathscr{K}(\Theta(E)),$$

which proves the theorem.

THEOREM 4. *Let*

$$0\to G\xrightarrow{\psi}E\to Q\to 0 \tag{8}$$

be an exact sequence of sheaves defined on X of the type described in §18, §19 *of Chap.* III. *Then*

$$\mathscr{K}(E)=\mathscr{K}(G)+\mathscr{K}(Q). \tag{9}$$

Proof. The proof is similar to that of the previous theorem, but we refer now to the results and notations of §18, §19 of Chap. III. On the graph $\mathscr{G}(\psi)$, we have

$$\psi_{;1}{}^*\Theta(E)=\psi_{;2}{}^*\Theta(G)+\Theta(S(\psi));$$

$\psi_{;1}$ (resp. $\psi_{;2}$) is an anti-monoidal transformation from $\mathscr{G}(\psi)$ onto $\mathscr{B}(E)$ resp. $\mathscr{B}(G)$); $\mathfrak{N}_\psi$ belongs to the divisor class $\Theta(S(\psi))$ and the restriction of $\psi_{;1}$ equips $\mathfrak{N}_\psi$ with the structure of the dual projective bundle of the sheaf $N(\mathscr{B}(E);\mathscr{B}(Q))$. Set $x=\psi_{;1}{}^*\Theta(E)$, $y=\psi_{;2}{}^*\Theta(G)$ and $s=\Theta(S(\psi))$. We can apply Theorem 1 since $s=x-y$ contains the non-singular member $\mathfrak{N}_\psi$ and we have

$$\mathscr{K}(x)=\mathscr{K}(y)+\mathscr{K}(\{x\}_{\mathscr{N}_\psi}). \tag{10}$$

The divisor class $\{x\}_{\mathscr{N}_\psi}$, which is the trace of x on the subvariety $\mathfrak{N}_\psi$, is the reciprocal image of the divisor class $\Theta(Q)$ with respect to the bundle projection from $\mathfrak{N}_\psi$ onto $\mathscr{B}(Q)$. From Theorem 2, we have

$$\mathscr{K}(x)=\mathscr{K}(\Theta(E)),$$
$$\mathscr{K}(y)=\mathscr{K}(\Theta(G)),$$
$$\mathscr{K}(\{x\}_{\mathscr{N}_\psi})=\mathscr{K}(\Theta(Q)),$$

so that (9) follows by substitution in (10).

§25. Consider the exact sequence

$$0\to F^{r-d}\xrightarrow{\psi^{r-d}}\cdots\to F^{s}\xrightarrow{\psi^{s}}F^{s-1}\to\cdots\to F^{0}\xrightarrow{\psi^{0}}Q\to 0 \tag{1}$$

of § 13, Chap. II; $F^{r-d}, \cdots, F^0$ are locally free sheaves defined on X; Q is the extension to X of a locally free sheaf defined on V, which sheaf we continue to denote as Q, where V is a non-singular subvariety of dimension d on X with $\dim. X = r$. The following theorem is of fundamental importance in the proof of the unicity of $\mathcal{A}$.

THEOREM 5. *For the exact sequence* (1), *we have*

$$\mathcal{K}(Q) = \sum_{l=0}^{r-d} (-1)^l \mathcal{K}(F^l). \tag{2}$$

The remainder of this § is devoted to proof of (2) which depends upon a lemma to be stated here but proved in the next §. X^* is the variety obtained from monoidal transformation of X centered on V; Φ is the anti-monoidal transformation from X^* onto X and V^* is the anti-center. Using the notations of § 13, we have the exact sequences

$$0 \to \mathrm{Im}[\psi_*{}^1] \to F_*{}^0 \to Q_* \to 0, \tag{3}$$

$$0 \to \mathrm{Im}[\psi_*{}^{s+1}] \to \mathrm{Ker}[\psi_*{}^s] \to Q_* \otimes \wedge^s(\delta N) \to 0 \tag{4}$$

for $1 \leqq s \leqq r-d-1$ and

$$0 \to \mathrm{Ker}[\psi_*{}^s] \to F_*{}^s \to \mathrm{Im}[\psi_*{}^s] \to 0 \tag{5}$$

for $1 \leqq s \leqq r-d$. We can apply Theorem 3 to (5) since (5) is an exact sequence of locally free sheaves defined on X^* and we obtain

$$\mathcal{K}(\mathrm{Im}[\psi_*{}^s]) = \mathcal{K}(F_*{}^s) - \mathcal{K}(\mathrm{Ker}[\psi_*{}^s]) \tag{6}$$

for $1 \leqq s \leqq r-d$. We can apply Theorem 4 to the exact sequences (4) and (3) since Q_* (resp. $Q_* \otimes \wedge^s(\delta N)$) is a locally free sheaf defined on the subvariety V^* of co-dimension 1 on X^*; thus we obtain

$$\mathcal{K}(Q_*) = \mathcal{K}(F_*{}^0) - \mathcal{K}(\mathrm{Im}[\psi_*{}^1]) \tag{7}$$

and

$$\mathcal{K}(Q_* \otimes \wedge^s(\delta N)) = \mathcal{K}(\mathrm{Ker}[\psi_*{}^s]) - \mathcal{K}(\mathrm{Im}[\psi_*{}^{s+1}]) \tag{8}$$

for $1 \leqq s \leqq r-d-1$. The following lemma will be proved in § 26.

LEMMA. $\mathcal{K}(Q_* \otimes \wedge^s(\delta N)) = 0$ *for* $1 \leqq s \leqq r-d-1$.

Consequently, we have

$$\mathcal{K}(\mathrm{Ker}[\psi_*{}^s]) = \mathcal{K}(\mathrm{Im}[\psi_*{}^{s+1}]) \tag{9}$$

for $1 \leqq s \leqq r-d$ so that we obtain from (6), (7) and (9)

$$\mathcal{K}(Q_*) = \sum_{l=0}^{r-d} (-1)^l \mathcal{K}(F_*{}^l) \tag{10}$$

since $\mathrm{Im}[\psi_*{}^{r-d}] = F_*{}^{r-d}$.

Applying Proposition 4 of § 24, we obtain $\mathcal{K}(F^l) = \mathcal{K}(F_*{}^l)$ since $F_*{}^l$ is the reciprocal image of F^l with respect to the anti-monoidal transformation Φ from X^* onto X; similarly, we have $\mathcal{K}(Q) = \mathcal{K}(Q_*)$ since the restriction of Φ to V^* equips V^* with the structure of the dual projective bundle of $N = N(X; V)$ and Q_* is the reciprocal image of Q with respect to that restriction mapping. Thus (10) gives

$$\mathcal{K}(Q) = \sum_{l=0}^{r-d} (-1)^l \mathcal{K}(F^l). \tag{2}$$

SUPPLEMENT. *For any resolution*

$$0 \to F^t \to \cdot \cdot \cdot \to F^0 \to Q \to 0 \tag{11}$$

of Q by locally free sheaves $F^0, \cdot \cdot \cdot, F^t$ on X (t is necessarily $\geqq r-d$ as follows from Chap. II), *we have*

$$\mathcal{K}(Q) = \sum_{l=0}^{t} (-1)^l \mathcal{K}(F^l).$$

§ 26. THEOREM 6. *Let E be a locally free sheaf of dimension n defined on the non-singular projective model X. Then we have*

$$\mathcal{A}(h\Theta(E) + \pi_E{}^*\vartheta) = \mathcal{A}(X)$$

for all $1 \leqq h \leqq n-1$, where $\pi_E{}^*\vartheta$ is the reciprocal image with respect to π_E of any divisor class ϑ on X.

The proof will be by induction on the dimension n of the sheaf E. The theorem is vacuously true for $n = 1$. Let $n > 1$, and, as the inductive hypothesis, assume that the theorem is true for all locally free sheaves of dimension less than n. The main part of the proof is given in two lemmas.

LEMMA 1. *Theorem 6 is true under the further assumption that E admits an exact sequence*

$$0 \to D \xrightarrow{\phi} E \to F \to 0, \tag{1}$$

of locally free sheaves defined on X, where D is a locally free sheaf of dimension one.

Proof. We shall apply the results of § 17, Chap. III to the exact sequence (1). We have dim. $\mathcal{B}(F) = \text{dim.}\, \mathcal{B}(E) - 1$ since D is a locally free sheaf of dimension one defined on X; this proves that $\phi_{;1}$ is a bi-regular mapping $\mathcal{B}(\phi)$ onto $\mathcal{B}(E)$ since $\phi_{;1}$ is an anti-monoidal transformation whose center is the subvariety $\mathcal{B}(F)$ on $\mathcal{B}(E)$. In fact, here the dual projective transformation $\mathcal{B}(\phi)$ is a regular rational mapping from $\mathcal{B}(E)$ onto $\mathcal{B}(D) = X$ and $\mathcal{B}(\phi)$ is equal to the composite mapping $\phi_{;2} \circ \phi_{;1}^{-1}$ where $\phi_{;1}^{-1}$ is the inverse map to $\phi_{;1}$. The projection $\phi_{;2}$ equips $\mathcal{B}(\phi)$ with the structure of the dual projective bundle of a locally free sheaf of dimension n defined on $\mathcal{B}(D) = X$. On $\mathcal{B}(\phi)$, we have

$$\phi_{;1}{}^*\Theta(E) = \phi_{;2}{}^*\Theta(D) + \Theta(S(\phi)), \tag{2}$$

where the divisor class $\Theta(S(\phi))$ contains the anti-center $\mathfrak{N}_\phi$, and of course $\phi_{;1}$ bi-regularly maps $\mathfrak{N}_\phi$ on $\mathcal{B}(F)$.

To avoid an unnecessarily pedantic notation, we shall identify $\mathcal{B}(\phi)$ with $\mathcal{B}(E)$ and $\mathfrak{N}_\phi$ with $\mathcal{B}(F)$ according to the mapping $\phi_{;1}$. Under this identification, the projection $\phi_{;2}$ is equal to π_E. Set $x = \phi_{;1}{}^*\Theta(E) = \Theta(E)$, $y = \phi_{;2}{}^*\Theta(D) = \pi_E{}^*\Theta(D)$, and $s = \Theta(S(\phi))$; hence s is the divisor class of $\mathcal{B}(F)$ on $\mathcal{B}(E)$ and we rewrite (2) as

$$x = y + s. \tag{2'}$$

The trace $\{x\}_{\mathcal{B}(F)}$ of x on $\mathcal{B}(F)$ is equal to $\Theta(F)$. From (2′), we have

$$hx + \pi_E{}^*\vartheta = (h-1)x + y + \pi_E{}^*\vartheta + s, \tag{3}$$

where h is any integer $\geqq 1$ and ϑ is any divisor class on X. We have

$$\begin{aligned} \mathfrak{a}(hx + \pi_E{}^*\vartheta) = \mathfrak{a}((h-1)x + y + \pi_E{}^*\vartheta) \\ + \mathfrak{a}(s) - \mathfrak{a}(\{(h-1)x + y + \pi_E{}^*\vartheta\}_{\mathcal{B}(F)}), \end{aligned} \tag{4}$$

as follows from Theorem 1 of § 22 since the divisor class s contains the non-singular member $\mathcal{B}(F)$.

Consider first the case where $h = 1$. Then (4) is simply

$$\mathfrak{a}(x + \pi_E{}^*\vartheta) = \mathfrak{a}(y + \pi_E{}^*\vartheta) + \mathfrak{a}(s) - \mathfrak{a}(\{y + \pi_E{}^*\vartheta\}_{\mathcal{B}(F)}). \tag{5}$$

Now $y + \pi_E{}^*\vartheta$ (resp. $\{y + \pi_E{}^*\vartheta\}_{\mathcal{B}(F)}$) is the reciprocal image of $\Theta(D) + \vartheta$ with respect to π_E (resp. π_F) so that, from Theorem 2,

$$\mathfrak{a}(y + \pi_E{}^*\vartheta) = \mathfrak{a}(\Theta(D) + \vartheta) = \mathfrak{a}(\{y + \pi_E{}^*\vartheta\}_{\mathcal{B}(F)}).$$

Thus we have

$$\mathcal{A}(x+\pi_E{}^*\vartheta)=\mathcal{A}(s);$$

but $\mathcal{A}(s)=\mathcal{A}(\mathcal{B}(F))$ since $\mathcal{B}(F)$ belongs to s, and, from the Fiber Law, $\mathcal{A}(\mathcal{B}(F))=\mathcal{A}(X)$; this proves that

$$\mathcal{A}(x+\pi_E{}^*\vartheta)=\mathcal{A}(X).$$

Now assume that $h>1$, $h\leqq n-1$ so that $1\leqq h-1\leqq n-2$. We have that F is a locally free sheaf of dimension $n-1$ defined on X and that

$$\{(h-1)x+y+\pi_E{}^*\vartheta\}_{\mathcal{B}(F)}=(h-1)\Theta(F)+\pi_F{}^*\Theta(D)+\vartheta);$$

hence, by our inductive hypothesis, we have

$$\mathcal{A}(\{(h-1)x+y+\pi_E{}^*\vartheta\}_{\mathcal{B}(F)})=\mathcal{A}(X),$$

and since $\mathcal{A}(s)=\mathcal{A}(\mathcal{B}(F))=\mathcal{A}(X)$, we obtain from (4) that

$$\mathcal{A}(hx+\pi_E{}^*\vartheta)=\mathcal{A}((h-1)x+y+\pi^*{}_E\vartheta).$$

Thus we have

$$\mathcal{A}(hx+\pi_E{}^*\vartheta)=\mathcal{A}(x+(h-1)y+\pi_E{}^*\vartheta), \tag{7}$$

and from the previous paragraph, we have that the right hand side of (7) is equal to $\mathcal{A}(X)$. The proof of the lemma is complete.

We turn to the case where E is an arbitrary locally free sheaf of dimension n defined on X.

Lemma 2. *Let $\Phi\colon Y\to X$ be a regular mapping from Y onto X which satisfies one of the conditions of the Fiber Law. If Theorem* 6 *is true for the reciprocal image sheaf ΦE of E with respect to Φ, then the theorem is true for the sheaf E.*

Proof. For we have that

$$h\Theta(\Phi E)+\pi_{\Phi E}{}^*(\Phi^*\vartheta) \tag{8}$$

is the reciprocal image of

$$h\Theta(E)+\pi_E{}^*\vartheta \tag{9}$$

with respect to the fiber preserving map from $\mathcal{B}(\Phi E)$ onto $\mathcal{B}(E)$ which covers the map $\Phi\colon Y\to X$ of the base spaces; the fiber preserving map satisfies one of the conditions of the Fiber Law since Φ is such a mapping; from Proposition 3 of § 22, we get that the virtual $\mathcal{A}$-genera of (8) and (9) are

equal. But if Theorem 6 is true for ΦE, then the virtual $\mathfrak{A}$-genus of (8) is equal to $\mathfrak{A}(Y)$, and, from the Fiber Law, $\mathfrak{A}(Y)=\mathfrak{A}(X)$, which proves Theorem 6 for the sheaf E.

Completion of the proof of Theorem 6: Consider the sheaf ${}^{\#}E$ of § 4, Chap. I; it is a locally free sheaf of dimension n defined on the dual projective bundle $\mathfrak{B}(\delta_{n-1}E)$ of the $(n-1)$-st derived sheaf $\delta_{n-1}E$ of E. The sheaf ${}^{\#}E$ is the reciprocal image of E with respect to the composite mapping

$$\Sigma=\pi_E\circ\pi_{\delta_1 E}\circ\cdot\cdot\cdot\circ\pi_{\delta_{n-2}E}\circ\pi_{\delta_{n-1}E}$$

from $\mathfrak{B}(\delta_{n-1}E)$ onto X. The residue class sheaf of ${}^{\#}E$ modulo the subsheaf $\delta_{n-1}E$ is a locally free sheaf of dimension $n-1$ defined on $\mathfrak{B}(\delta_{n-1}E)$, as follows from the composition series (4) of § 4, Chap. I. Thus Theorem 6 is true for ${}^{\#}E$ since ${}^{\#}E$ satisfies the addition assumption of Lemma 1. But Σ is composed of bundle projections of dual projective bundle so that by repeated application of Lemma 2, Theorem 6 is true for E since it is true for ${}^{\#}E$.

THEOREM 7. *E, F are locally free sheaves defined on X, and E is of dimension $n>1$. Then we have*

$$\mathcal{K}(\pi_E F\otimes B^{-h}(E))=0$$

for all $1\leqq h\leqq n-1$; ($B^{-1}(E)$ is such that $B^{-1}(E)\otimes B(E)=\mathfrak{O}_{\mathfrak{B}(E)}$ and $B^{-h}(E)$ is the tensor product of $B^{-1}(E)$ with itself h times).

Proof. Let T equal the dual projective bundle of the sheaf $\pi_E F$ defined on $\mathfrak{B}(E)$; T is the induced fiber bundle of $\mathfrak{B}(F)$ with respect to the mapping $\pi_E\colon\mathfrak{B}(E)\to X$. Let $\xi_1\colon T\to\mathfrak{B}(E)$ denote the bundle projection of T, and let $\xi_2\colon T\to\mathfrak{B}(F)$ denote the fiber preserving map which covers the mapping $\pi_E\colon\mathfrak{B}(E)\to X$ of the base spaces of T and $\mathfrak{B}(F)$. It follows by inspection that ξ_2 equips T with the structure of the dual projective fiber bundle $\mathfrak{B}(\pi_F E)$ of the sheaf $\pi_F E$ and that ξ_1 is the fiber preserving map from $T=\mathfrak{B}(\pi_F E)$ onto $\mathfrak{B}(E)$ which covers the mapping $\pi_F\colon\mathfrak{B}(F)\to X$ of the base spaces of $\mathfrak{A}(\pi_F E)$ and $\mathfrak{B}(E)$. Thus we have that

$$\Theta(\pi_E F)=\xi_2{}^*\Theta(F),$$
$$\Theta(\pi_F E)=\xi_1{}^*\Theta(E).$$

The basic class of the sheaf $\pi_E F\otimes B^{-h}(E)$ is the divisor class

$$\Theta(\pi_E F\otimes B^{-h}(E))=\Theta(\pi_E F)+\xi_1{}^*\Theta(B^{-h}(E))$$

on T, as follows from § 8 of Chap. I. But

$$\xi_1{}^*\Theta(B^{-h}(E)) = -h\xi_1{}^*\Theta(B(E)) = -h\xi_1{}^*\Theta(E),$$

which gives

$$\Theta(\pi_E F \otimes B^{-h}(E)) = \xi_2{}^*\Theta(F) - h\xi_1{}^*\Theta(E). \tag{10}$$

For $1 \leqq h \leqq n-1$, we can apply Theorem 6 to the sheaf $\pi_F E$ and obtain

$$\mathcal{A}(h\xi_1{}^*\Theta(E) - \xi_2{}^*\Theta(F)) = \mathcal{A}(\mathcal{B}(F));$$

this is permissible since $\xi_1{}^*\Theta(E) = \Theta(\pi_F E)$, and $\pi_F E$ is a locally free sheaf of dimension n defined on $\mathcal{B}(F)$. This proves, in virtue of (10), that

$$\mathcal{A}(-\Theta(\pi_E F \otimes B^{-h}(E))) = \mathcal{A}(\mathcal{B}(F));$$

but

$$\mathcal{K}(\pi_E F \otimes B^{-h}(E)) = \mathcal{A}(\mathcal{B}(E)) - \mathcal{A}(-\Theta(\pi_E F \otimes B^{-h}(E)))$$

and since $\mathcal{A}(\mathcal{B}(E)) = \mathcal{A}(X) = \mathcal{A}(\mathcal{B}(F))$, our theorem follows.

The unproved lemma of § 25 is an immediate consequence of the following more general result.

THEOREM 8. *E, F are locally free sheaves as defined on X, and E is of dimension $n > 1$. Then*

$$\mathcal{K}(\pi_E F \otimes \wedge^s(\delta E)) = 0$$

for all $1 \leqq s \leqq n-1$; (recall that $\wedge^s(\delta E)$ is the s-fold exterior product of the derived sheaf δE of the sheaf E).

Proof. We rewrite the exact sequence (7) of § 5, Chap. I:

$$0 \wedge^n(\pi_E E) \otimes B^{-(n-s)}(E) \to \cdots \to \wedge^{s+1}(\pi_E E) \otimes B^{-1}(E) \to \wedge^s(\delta E) \to 0.$$

It is an exact sequence of locally free sheaves defined on $\mathcal{B}(E)$ and it is composed of the exact sequences

$$0 \to \wedge^{s+k}(\delta E) \otimes B^{-k}(E) \to \wedge^{s+k}(\pi_E E) \otimes B^{-k}(E) \to \wedge^{s+k-1}(\delta E) \otimes B^{-(k-1)}(E) \to 0 \tag{11}$$

for $1 \leqq k \leqq n-s$. Tensorizing (11) with $\pi_E F$ preserves exactness since $\pi_E F$ is a locally free sheaf, and we obtain the exact sequences

$$0 \to \pi_E F \otimes \wedge^{s+k}(\delta E) \otimes B^{-k}(E) \to \pi_E F \otimes \wedge^{s+k}(\pi_E E) \otimes B^{-k}(E) \to \pi_E F \otimes \wedge^{s+(k-1)}(\delta E) \otimes B^{-(k-1)}(E) \to 0 \tag{12}$$

for $1 \leqq k \leqq n-s$. From Theorem 3, we obtain

$$\mathcal{K}(\pi_E F \otimes \wedge^s(\delta E)) = \sum_{k=1}^{n-s} (-1)^{k-1} \mathcal{K}(\pi_E F \otimes \wedge^{s+k}(\pi_E E) \otimes B^{-k}(E)) \tag{13}$$

since $\wedge^n(\delta E)$ is the sheaf zero (δE is of dimension $n-1$). On the other hand,

$$\pi_E F \otimes \wedge^{s+k}(\pi_E E) = \pi_E(F \otimes \wedge^{s+k}(E))$$

and from Theorem 7, we have

$$\mathcal{K}(\pi_E(F \otimes \wedge^{s+k}(E)) \otimes B^{-k}(E)) = 0$$

for all $1 \leqq k \leqq n-s$; thus we finally have

$$\mathcal{K}(\pi_E F \otimes \wedge^s(\delta E)) = 0.$$

§27. $K[X_0, \cdots, X_r]$ is the ring of polynomials in the $r+1$ indeterminates $X_0, \cdots, X_r$ with coefficients in the fixed field K. By an S-module (terminology of Koszul, Serre, $\cdots$), we mean simply a module $\mathfrak{M}$ over the ring $S = K[X_0, \cdots, X_r]$. All S-modules considered here are unitary, graded, and finitely generated by homogeneous elements.

Unitary: this means $1 \cdot m = m$ for all $m \in \mathfrak{M}$, where 1 is the constant polynomial 1 in $K[X_0, \cdots, X_r]$.

Graded: $\mathfrak{M}$ is equal to a direct sum of additive subgroups $\mathfrak{M}^k$, $0 \leqq k < \infty$; the elements of $\mathfrak{M}^k$ are called homogeneous of degree k and we require that $Fm \in \mathfrak{M}^{h+k}$ for a homogeneous polynomial F of degree h and $m \in \mathfrak{M}^k$.

Finitely generated by homogeneous elements: it is possible to choose finitely many homogeneous elements $m_1, \cdots, m_t$ from $\mathfrak{M}$ which generate $\mathfrak{M}$ as module over $K[X_0, \cdots, X_r]$; this implies that $\mathfrak{M}^k$ is a finite dimensional vector space over the field K; $m_1, \cdots, m_t$ is called a system of generators of $\mathfrak{M}$.

By a homomorphism $\Psi: \mathfrak{M} \rightarrow \mathfrak{N}$ of S-modules $\mathfrak{M}$, $\mathfrak{N}$, we shall mean a homomorphism of their structure as modules over $K[X_0, \cdots, X_r]$ that preserves degree; that is Ψ maps $\mathfrak{M}^k$ into $\mathfrak{N}^k$ for all $0 \leqq k < \infty$. It is easy to check that $\mathrm{Ker}[\Psi: \mathfrak{M} \rightarrow \mathfrak{N}]$ and $\mathrm{Im}[\Psi: \mathfrak{M} \rightarrow \mathfrak{N}]$ are sub S-modules of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. $\mathfrak{M}$ is called a "free" S-module if we can choose a (homogeneous) system of generators of $\mathfrak{M}$ which generate $\mathfrak{M}$ as a free module over $K[X_0, \cdots, X_r]$.

Let $\mathfrak{M}$ be any S-module. Since $\mathfrak{M}$ is finitely generated, it is possible to choose a free S-module $\mathfrak{F}_0$ and a homomorphism Ψ_0 from $\mathfrak{F}_0$ onto $\mathfrak{M}$ so that we have the exact sequence of S-modules

$$0 \rightarrow \mathrm{Ker}[\Psi_0] \rightarrow \mathfrak{F}_0 \rightarrow \mathfrak{M} \rightarrow 0.$$

Repeating this process several times, we arrive at an exact sequence of S-modules

$$\mathfrak{F}_t \xrightarrow{\Psi_t} \mathfrak{F}_{t-1} \to \cdots \to \mathfrak{F}_0 \to \mathfrak{M} \to 0,$$

where $\mathfrak{F}_0, \cdots, \mathfrak{F}_t$ are free S-modules. Hilbert's famous theorem on "chains of syzygies" asserts that $\operatorname{Ker}[\Psi_t]$ is a free S-module if $t \geqq r$. Thus, for any S-module $\mathfrak{M}$, it is possible to construct an exact sequence

$$(1) \qquad 0 \to \mathfrak{F}_r \xrightarrow{\Psi_r} \mathfrak{F}_{r-1} \to \cdots \xrightarrow{\Psi_1} \mathfrak{F}_0 \xrightarrow{\Psi_0} \mathfrak{M} \to 0,$$

where $\mathfrak{F}_0, \cdots, \mathfrak{F}_r$ are free S-modules.

For any h, $0 \leqq h < \infty$, we have the exact sequence of vector spaces

$$0 \to \mathfrak{F}_r^h \to \cdots \to \mathfrak{F}_s^h \to \cdots \to \mathfrak{F}_0^h \to \mathfrak{M}^h \to 0,$$

where $\mathfrak{M}^h$ (resp. $\mathfrak{F}_s^h$) is the vector space over K consisting of the homogeneous elements of degree h of $\mathfrak{M}$ (resp. $\mathfrak{F}_s$). For the dimensions of these vector spaces, we have the relation

$$\dim. \mathfrak{M}^h = \sum_{s=0}^{r} (-1)^s \dim. \mathfrak{F}_s^h.$$

Now for all $h \gg 0$ (that is for all sufficiently large h), we have

$$\dim. \mathfrak{F}_s^h = \chi(\mathfrak{F}_s; h),$$

where $\chi(\mathfrak{F}_s; h)$ is a polynomial in h with rational coefficients; this is obvious if we recall that each $\mathfrak{F}_s$ is a free S-module. Set

$$\chi(\mathfrak{M}; h) = \sum_{s=0}^{r} (-1)^s \chi(\mathfrak{F}_s; h).$$

Then $\chi(\mathfrak{M}; h)$ is a polynomial in h with rational coefficients and we have that for $h \gg 0$,

$$\dim. \mathfrak{M}^h = \chi(\mathfrak{M}; h).$$

$\chi(\mathfrak{M}; h)$ is called the Hilbert characteristic polynomial of the S-module $\mathfrak{M}$; and we have repeated Hilbert's own proof of its existence.

§ 28. P^r $(= P)$ is a projective space of dimension r, and let us identify $K[X_0, \cdots, X_r]$ with the homogeneous coordinate ring over K of P. Serre has given a method whereby one associates with any S-module $\mathfrak{M}$ a (coherent, algebraic) sheaf M defined on P. We shall review his construction first for the special case of a free S-module $\mathfrak{F}$. The associated sheaf F is then a

locally free sheaf defined on P. For let $f_1, \cdots, f_n$ freely generate $\mathfrak{F}$, so that $f_1, \cdots, f_n$ are homogeneous elements say of degrees $h_1, \cdots, h_n$ respectively; let $\{U_\alpha\}$, $\alpha = 0, 1, \cdots, r$, be the standard covering of P, that is U_α is the open set on P whose frontier is the hyperlane $X_\alpha = 0$. The restriction of F to any U_α is a free sheaf generated by sections $f^\alpha{}_1, \cdots, f^\alpha{}_n$ of F over U_α; on $U_\alpha \cap U_\beta$, we have the transition laws

$$f^\alpha{}_i = (X_\beta / X_\alpha)^{h_i} f^\beta{}_i, \qquad 1 \leqq i \leqq n,$$

(h_i is the degree of the generator f_i). It is evident that the associated sheaf F is equal to the direct sum of the sheaves $\mathcal{O}(-h_1), \cdots, \mathcal{O}(-h_n)$. ($\mathcal{O}(-h)$ is isomorphic to the sheaf $\mathcal{L}(\vartheta_h)$ of germs of rational functions on P which are multiples of a divisor ϑ_h belonging to the linear system of hypersurface sections of order h on P.)

A locally free sheaf defined on P is called "S-free" if it is the associated sheaf of a free S-module; it is clear that a sheaf is S-free if and only if it is a direct sum of finitely many sheaves of type $\mathcal{O}(-h)$ with $h \geqq 0$.

Let $\mathfrak{G}$ be a free S-module, and let there be a homomorphism $\Psi \colon \mathfrak{G} \to \mathfrak{F}$. Then there is an associated sheaf homomorphism $\psi \colon G \to F$ of the associated S-free sheaves. Let Ψ be described by

$$\Psi \colon g_p \to \sum_{i=1}^{n} A_{i,p} f_i, \qquad 1 \leqq p \leqq m$$

(assume that $\mathfrak{G}$ has m generators), where the A's are homogeneous polynomials and

$$\deg . g_p = \deg . A_{i,p} + \deg . f_i$$

for all $1 \leqq i \leqq n$. The restriction of the sheaf homomorphism $\psi \colon G \to F$ to any U_α is described by

$$g_p{}^\alpha \to \sum_{i=1}^{n} (A_{i,p} / (X_\alpha)^{\nu_{i,p}}) f^\alpha{}_i, \qquad 1 \leqq p \leqq m,$$

where $\nu_{i,p} = \deg . A_{i,p}$.

Let $\mathfrak{M}$ be any S-module, and choose free S-modules $\mathfrak{F}$, $\mathfrak{G}$ and homomorphisms $\Psi' \colon \mathfrak{F} \to \mathfrak{M}$, $\Psi \colon \mathfrak{G} \to \mathfrak{F}$ such that

$$\mathfrak{G} \xrightarrow{\Psi} \mathfrak{F} \xrightarrow{\Psi'} \mathfrak{M} \to 0$$

is an exact sequence of S-modules. The associated sheaf M of $\mathfrak{M}$ is defined to be the residue class sheaf $F/\mathrm{Im}[\psi]$, where $\mathrm{Im}[\psi]$ is the image of G by the

associated homomorphism $\psi\colon G \to F$; it is easy to check that M depends solely on $\mathfrak{M}$ and that there is an associated sheaf homomorphism $\psi'\colon F \to M$ so that we have the exact sequence of sheaves

$$G \xrightarrow{\psi} F \xrightarrow{\psi'} M \to 0.$$

For an arbitrary S-module $\mathfrak{M}$, we recopy the "chain of syzygies" (1) of the preceding §:

$$(1) \qquad 0 \to \mathfrak{F}_r \xrightarrow{\Psi_r} \cdots \xrightarrow{\Psi_1} \mathfrak{F}_0 \xrightarrow{\Psi_0} \mathfrak{M} \to 0.$$

Applying Serre's construction, we take associated sheaves and homomorphisms and we have the exact sequence of sheaves

$$(2) \qquad 0 \to F_r \xrightarrow{\psi_r} \cdots \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} M \to 0,$$

where $F_0, \cdots, F_r$ are S-free sheaves. According to Serre's cohomology theory, we have the Euler-Poincaré characteristic $\chi(P, M)$ of the sheaf M, where

$$\chi(P, M) = \sum_{t=0}^{\infty} (-1)^t \dim. H^t(P, M)$$

($H^t(P, M)$ is the t-dimensional cohomology module of the sheaf M defined on P). We also have that

$$(3) \qquad \chi(P, M) = \sum_{s=0}^{r} (-1)^s \chi(P, F_s).$$

Incidentally, Serre has proved that

$$\chi(P, M) = \chi(\mathfrak{M}, 0),$$

that is, where $\chi(\mathfrak{M}, 0)$ is the value of the Hilbert characteristic polynomial of $\mathfrak{M}$ for $h = 0$.

Let V be a non-singular subvariety on P, and let $\mathfrak{R}(V)$ be the homogeneous coordinate ring of V, that is $\mathfrak{R}(V)$ is the residue class ring of $K[X_0, \cdots, X_r]$ modulo the ideal determined by V. Then $\mathfrak{R}(V)$ is an S-module. The associated sheaf of $\mathfrak{R}(V)$ is easily seen to be the sheaf $\mathcal{O}_V$ of local rings of V extended to P. From (2), we get, upon setting $M = \mathcal{O}_V$, the exact sequence

$$(4) \qquad 0 \to F_r \xrightarrow{\psi_r} \cdots \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} \mathcal{O}_V \to 0,$$

where $F_0, \cdots, F_r$ are S-free sheaves. From the Supplement to Theorem 5 of § 25, we obtain that

$$\mathcal{K}(\mathcal{O}_V) = \sum_{s=0}^{r} (-1)^s \mathcal{K}(F_s). \tag{5}$$

In the next §, we shall prove that $\mathcal{K}(F) = \chi(P, F)$ for an S-free sheaf F. On the basis of this assertion, we obtain from (3) and (5) that

$$\chi(V, \mathcal{O}_V) = \mathcal{K}(\mathcal{O}_V).$$

Since $\mathcal{K}(\mathcal{O}_V) = \mathfrak{a}(V)$, we have proved the unicity $\mathfrak{a}$, for we now have:

THEOREM 9. *Let V be a non-singular projective model. Then we have*

$$\mathfrak{a}(V) = \chi(V, \mathcal{O}_V).$$

§ 29. PROPOSITION 5. *Let F be an S-free sheaf. Then $\mathcal{K}(F) = \chi(P; F)$.* (This proposition completes the proof of Theorem 9.)

Proof. F, since it is S-free, is equal to a direct sum

$$\mathcal{O}(-h_1 + \cdots + \mathcal{O}(-h_n)$$

with each $h_i \geqq 0$. Now, from the Serre cohomology theory, we have

$$\chi(P; F) = \sum_{t=1}^{n} \chi(P; \mathcal{O}(-h_t)),$$

and, from Theorem 3, we have

$$\mathcal{K}(F) = \sum_{t=1}^{n} \mathcal{K}(\mathcal{O}(-h_t)).$$

Hence it suffices to prove our proposition merely for sheaves of type $\mathcal{O}(-h)$, $h \geqq 0$.

Proposition 5 follows from:

PROPOSITION 6. $\mathcal{K}(\mathcal{O}(-h)) = \chi(P, \mathcal{O}(-h)) \quad (h \geqq 0)$.

Proof. The proof is by induction on the dimension r of P and the degree h. We assume that our proposition is true for S-free sheaves of dimension one defined on projective spaces of dimension less than r. (It is clearly true for a projective space of dimension zero.) For the case $h = 0$ on P, it is true; (it is well known that

$$\chi(P; \mathcal{O}_P) = 1,$$

$\mathcal{O}_P = \mathcal{O}(0)$, and $\mathcal{K}(\mathcal{O}_P) = \mathcal{A}(P) = 1$ by the Normalization Axiom). Therefore, we assume that $h > 0$ and that our proposition is true for $h-1$. Consider the exact sequence

$$0 \to \mathcal{O}(-h) \to \mathcal{O}(-(h-1)) \to \mathcal{O}_{P'}(-(h-1)) \to 0;$$

P' is a projective subspace of dimension $r-1$ on P and $\mathcal{O}_{P'}(-(h-1))$ is the extension to P of an S-free sheaf defined on P'. From the Serre cohomology theory, we have

$$(1) \qquad \chi(P; \mathcal{O}(-h)) = \chi(P; \mathcal{O}(-(h-1))) - \chi(P', \mathcal{O}_{P'}(-(h-1))),$$

and from Theorem4, we have

$$(2) \qquad \mathcal{K}(\mathcal{O}(-h)) = \mathcal{K}(\mathcal{O}(-(h-1))) - \mathcal{K}(\mathcal{O}_{P'}(-(h-1))).$$

From our inductive assumption, we obtain that the right hand sides of (1) and (2) are equal, which proves

$$\mathcal{K}(\mathcal{O}(-h)) = \chi(P; \mathcal{O}(-h)).$$

THE JOHNS HOPKINS UNIVERSITY.

A DUALITY THEORY FOR INJECTIVE MODULES.*

(Theory of Quasi-Frobenius Modules)

By Goro Azumaya.[1]

Introduction. Let A be a finite-dimensional algebra with identity element over a field Φ. Let M be a finitely generated left A-module. Then M, when regarded as a left representation space, defines a representation of A in Φ, and there corresponds to this representation a finitely generated right A-modul M^*, which is called the dual representation space of M. M^* is nothing but the conjugate space of M, i. e., the vector space consisting of all Φ-linear mappings f of M into Φ, where fa, $a \in A$, is defined to be the mapping $x \rightarrow f(ax)$, $x \in M$, and moreover we have, by associating M with M^*, a one-to-one correspondence (up to isomorphisms) between finitely generated left and right A-modules. The present paper establishes a theory which extends this known situation to the case where A is a ring with minimum condition and possessing a certain type of injective module, so that it provides also a generalization of the theory of quasi-Frobenius rings, which has been developed mainly by T. Nakayama, M. Hall and M. Ikeda. Our principal results are summarized as follows:

Let A be a ring with identity element and satisfying the left minimum condition.[2] Suppose that Q is a finitely generated left A-module which is injective and contains an isomorphic image of every irreducible left A-module. Let A^* be the A-endomorphism ring of Q, considered as a right operator-ring. Then we have first that the same situations hold quite symmetrically for the A^*-module Q, that is, A^* satisfies the right minimum condition, Q is, as right A^*-module, both finitely generated and injective and contains an isomorphic image of every irreducible right A^*-module, and moreover A coincides with the A^*-endomorphism ring of Q (Theorem 6). Now, let M be a left A-module. We consider the module $M^* = \mathrm{Hom}_A(M, Q)$ consisting of all

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[2] By left minimum (or maximum) condition we mean the minimum (or maximum) condition for left ideals. As is well-known, the left minimum condition implies always the left maximum condition, under the assumption of the existence of identity element.

A-homomorphisms f of M into Q. M^* can be converted into a right A^*-module by defining fa^*, $a^* \in A^*$, to be the mapping $x \rightarrow f(x)a^*$, $x \in M$, which we shall call the dual module of M (with respect to Q).[3] Similarly, we may define a dual module for every right A^*-module. We have then the following fundamental duality theorem: Let M be a finitely generated left A-module. Then its dual module M^* is also finitely generated, and moreover, M may be regarded as the dual module of M^* in the natural way. The same holds also for every finitely generated right A^*-module (Theorem 8). Let now M_1 and M_2 be two finitely generated left A-modules and let M_1^* and M_2^* be their respective dual modules. Then with each A-homomorphism ϕ of M_1 into M_2 we can associate, in the usual manner, its dual mapping ϕ^* which is an A^*-homomorphism of M_2^* into M_1^*, and in this case, ϕ is regarded, by virtue of the above duality theorem, as the dual mapping of ϕ^*. We have thus an isomorphism $\phi \leftrightarrow \phi^*$ between the groups $\mathrm{Hom}_A(M_1, M_2)$ and $\mathrm{Hom}_{A^*}(M_2^*, M_1^*)$. Further, it is clear that if ψ is an A-homomorphism of M_2 into a third finitely generated left A-module, then $(\psi\phi)^* = \phi^*\psi^*$.

Interesting is however the fact that a complete converse of these situations holds in the following form: Let A and A^* be two rings with identity elements (but not be assumed to satisfy any chain conditions). Suppose that there is associated with each finitely generated left A-module M a finitely generated right A^*-module M^* so that the M^*'s exhaust, up to isomorphisms, all finitely generated right A^*-modules. Suppose furthermore that for each pair of finitely generated left A-modules M_1 and M_2 there is given an isomorphism $\phi \leftrightarrow \phi^*$ of $\mathrm{Hom}_A(M_1, M_2)$ onto $\mathrm{Hom}_{A^*}(M_2^*, M_1^*)$ in such a way that these isomorphisms together fulfill $(\psi\phi)^* = \phi^*\psi^*$. Then A and A^* indeed satisfy the left and right minimum conditions respectively, and we can find a two-sided A-A^*-module Q such that Q, when regarded as a left A-module, is of the same type as above and A^* coincides with its A-endomorphism ring, and moreover every M^* may be so identified with the dual module of M with respect to Q that every ϕ^* coincides with the dual mapping of ϕ (Theorem 10).

In order to establish the above results, it is indispensable to make use of an important concept of quasi-Frobenius two-sided modules for two rings A and A^*, which was however essentially introduced in the recent paper of Morita and Tachikawa [9], and in fact our theory should also be regarded as a theory of such modules. In particular, [9, Theorem 1.1], in a generalized

[3] Here and hereafter, one should pay attention to the fact that A^* is not necessarily (isomorphic to) the dual module of the left A-module A. Whenever there is a fear of such confusion, we shall use the notation A_s for the left (or right) A-module A.

form, plays a basic role. It turns out, among other things, that under the respective assumptions of the left and the right minimum conditions for A and A^*, quasi-Frobenius modules are exactly the same as the above considered modules Q, when regarded as two-sided A-A^*-modules, and moreover a ring A satisfying the left or the right minimum condition is a quasi-Frobenius ring if and only if it is quasi-Frobenius as a two-sided A-module. On the other hand, it may be of some interest that the notion of quasi-Frobenius modules gives a natural extension of the density theorem for irreducible modules and completely reducible modules. In fact, it will be shown in particular that a necessary and sufficient condition for a two-sided A-A^*-module, which is both faithful and completely reducible with respect to A, to be quasi-Frobenius is that A^* is a dense subring of the A-endomorphism ring of the module. Furthermore, we shall, in the last section, apply the above theory to algebras over a commutative ring with minimum condition to make it possible, for instance, to verify for the first time that the symmetricity of an algebra over a field is entirely independent of the choice of the base field.

It should be noted, in this connection, that a ring A with the left minimum conditions does not always possess an injective left module Q of the above mentioned type, as has recently been shown by Rosenberg and Zelinsky [14], while every finite-dimensional algebra A as well as every quasi-Frobenius ring A certainly has such a module.

Needless to say, the present study is indebted much to the fundamental works of Nakayama [11, 12] and Ikeda [5]. Also, I wish to express my thanks to Professor N. Jacobson who let me have an opportunity to discuss fully about the present subject in his seminar, as well as to Professors A. Rosenberk and D. Zelinsky who have communicated to me valuable information on the existence of injective modules during the preparation of this paper.

1. **Preliminaries.** Throughout this paper we shall assume, unless otherwise stated, that all rings considered have identity elements and also all (left, right, or two-sided) modules over rings are unital in the sense that identity elements operate on the modules as identity (cf. Jacobson [7, p. 1]).

Let A be a ring. A left A-module Q is, following Cartan and Eilenberg [1], called *injective* with respect to A (or A-injective) if given an left A-module M, an A-submodule M' and an A-homomorphism $f': M' \to Q$, there is an extension A-homomorphism $f: M \to Q$. We may however restrict our-

selves here to the particular case where $M = A$, as a matter of fact ([1, Theorem I. 3. 2]). Now, in order that Q be injective it is necessary and sufficient that Q be a direct summand in every extension A-module. In fact, we have more strongly the following

PROPOSITION 1. *A left A-module Q is injective if (and only if) Q is a direct summand in every extension A-module which is expressible as a sum of Q and a cyclic A-submodule.*

Proof. Let $\mathfrak{l}$ be a left ideal of A and f an A-homomorphism of $\mathfrak{l}$ into Q. Let $M = A \oplus Q$ be the direct sum of two left A-modules A and Q. Then all elements of the form $(a, -f(a))$ with $a \in \mathfrak{l}$ constitute an A-submodule M' of M. We consider the factor module $M'' = M - M'$. By identifying $u \in Q$ with the coset of $(0, u)$ modulo M', Q is clearly imbedded isomorphically into M''. Furthermore, A is, by associating $a \in A$ with the coset of $(a, 0)$, mapped homomorphically onto a (cyclic) submodule of M'', and so M'' is a sum of Q and this submodule. Consequently, M'' may be expressed as a direct sum of Q and a second submodule. Let v be the Q-component of the coset of $(1, 0)$ relative to this direct decomposition. Then we have immediately $av = f(a)$ for every $a \in \mathfrak{l}$, which shows that Q is injective.

COROLLARY. *If Q is a finitely generated left A-module, then Q is injective if and only if Q is a direct summand in every finitely generated extension A-module of Q.*

Now let Q be any left A-module. Consider a maximal left ideal $\mathfrak{l}$ of A. One can easily see that Q contains a minimal A-submodule which is A-isomorphic to the irreducible factor module $A - \mathfrak{l}$ if and only if the right annihilator $r_Q(\mathfrak{l})$ of $\mathfrak{l}$ in Q is non-zero, and in fact, when this is the case, cyclic submodules Au with non-zero elements u in $r_Q(\mathfrak{l})$ and only those are isomorphic images of $A - \mathfrak{l}$ in Q. On the other hand, if $\mathfrak{m}$ is an irreducible A-submodule of Q and if u is any non-zero element in $\mathfrak{m}$ then the left annihilator $l(u) = l_A(u)$ of u in A is a maximal left ideal of A and $\mathfrak{m} = Au$ is isomorphic to $A - l(u)$. These facts mean that the (A-)socle of Q, i.e., the sum of all irreducible A-submodules of Q coincides with the sum of the $r_Q(\mathfrak{l})$'s for all maximal left ideals $\mathfrak{l}$ of A. Moreover, for any given maximal left ideal $\mathfrak{l}$ of A such that Q contains an isomorphic image of $A - \mathfrak{l}$, the homogeneous component of the socle belonging to the irreducible left A-module $A - \mathfrak{l}$ (cf. [7, p. 63]) may be expressed not only in the form $Ar_Q(\mathfrak{l})$ but also as the sum of the $r_Q(\mathfrak{l}')$'s for all those maximal left ideals $\mathfrak{l}'$

for which $A - \mathfrak{l}'$ is isomorphic to $A - \mathfrak{l}$. We now call Q *distinguished* with respect to A (or A-distinguished) if Q contains an isomorphic image of every irreducible left A-module, or equivalently, if $r_Q(\mathfrak{l}) \neq 0$ for all maximal left ideals $\mathfrak{l}$ of A. Because of the fact that every proper left ideal is contained in a maximal left ideal, it is clear that Q is distinguished if and only if $r_Q(\mathfrak{l}) \neq 0$ for all proper left ideals $\mathfrak{l}$ of A. Further, we shall call Q *weakly distinguished* if, for any A-submodules $\mathfrak{m}$ and $\mathfrak{m}'$ of Q such that $\mathfrak{m} \supset \mathfrak{m}'$ and the factor module $\mathfrak{m} - \mathfrak{m}'$ is irreducible, Q contains an isomorphic image of $\mathfrak{m} - \mathfrak{m}'$. Evidently, distinguishedness implies weak distinguishedness.

Finally, we shall mean by the *capacity* of any irreducible left A-module the (finite or infinite) dimension of it over its endomorphism division ring.

2. Quasi-Frobenius modules. We consider, besides A, a second ring A^*, and suppose that Q is a two-sided A-A^*-module (in the sense that Q is a left A- and a right A^*-module at the same time and $(au)a^* = a(ua^*)$ for every $a \in A$, $u \in Q$, $a^* \in A^*$). Let M be a left A-module, and let $M^* = \mathrm{Hom}_A(M, Q)$ be the module consisting of all A-homomorphisms of M into Q. For any $x \in M$ and $f \in M^*$, we denote by xf the image of x by f. M^* can be made into a right A^*-module by setting $x(fa^*) = (xf)a^*$, $a^* \in A^*$, which we shall call the *right-dual module* of M with respect to Q. Similarly, we may define a *left-dual module* for any right A^*-module. Now, we call Q a *quasi-Frobenius two-sided A-A^*-module* if i) Q is faithful (with respect to both A and A^*), and ii) for every maximal left ideal $\mathfrak{l}$ of A and for every maximal right ideal $\mathfrak{r}$ of A^* the right annihilator $r_Q(\mathfrak{l})$ and the left annihilator $l_Q(\mathfrak{r})$ of $\mathfrak{l}$ and $\mathfrak{r}$ in Q are A^*-irreducible and A-irreducible respectively unless they are zero. If we observe however that for any left ideal $\mathfrak{l}$ of A the right annihilator $r_Q(\mathfrak{l})$ may be regarded as the right-dual module of $A - \mathfrak{l}$ and the similar holds for every right ideal, we may evidently replace the second condition ii) by the following: ii′) the right-dual module of every irreducible left A-module as well as the left-dual module of every irreducible right A^*-module, both with respect to Q, is irreducible whenever it is non-zero (or equivalently, whenever Q contains an isomorphic image of the given irreducible module).

THEOREM 1. *Let Q be a quasi-Frobenius two-sided A-A^*-module. Then the A-socle of Q coincides with the A^*-socle of Q. Moreover, if M is an irreducible left A-module such that Q contains an isomorphic image of M, or equivalently, the right-dual module M^* of M is irreducible, then the homogeneous (A-)component of the common socle belonging to M and the*

homogeneous (A^*-)*component belonging to* M^* *coincide and it is a minimal* A-A^*-*submodule.*

Proof. The homogeneous A-component of the A-socle belonging to M is the sum of the $r_Q(\mathfrak{l})$'s for all those maximal left ideals $\mathfrak{l}$ of A for which $A-\mathfrak{l}$ are isomorphic to M (§1). Such $r_Q(\mathfrak{l})$'s are however all (A^*-)isomorphic to M^*, and so the A-component is contained in the homogeneous A^*-component of the A^*-socle belonging to M^*. Since M is obviously the left-dual of M^*, we can conclude by symmetry that both the components coincide. Now, for a fixed maximal left ideal $\mathfrak{l}$ as above, any non-zero A-submodule of the common component contains a non-zero element of $r_Q(\mathfrak{l})$, because it contains an isomorphic image of $M \cong A-\mathfrak{l}$. This, together with the A^*-irreducibility of $r_Q(\mathfrak{l})$, implies that any non-zero A-A^*-submodule of the component contains $Ar_Q(\mathfrak{l})$, whereas the latter coincides with the component itself (§1), which shows nothing but the minimality of the component as a two-sided A-A^*-module. The coincidence of socles follows immediately from that of homogeneous components.

Let M be a left A-module and M^* a right A^*-module. Suppose that for any $x \in M$ and $y \in M^*$ there is defined a product xy in Q satisfying the following conditions, for $x, x' \in M$, $y, y' \in M^*$, $a \in A$, $a^* \in A^*$:

$$(x+x')y = xy + x'y, \qquad x(y+y') = xy + xy'$$
$$(ax)y = a(xy), \qquad (xy)a^* = x(ya^*).$$

If moreover $xM^* = 0$, $x \in M$, and $My = 0$, $y \in M^*$, imply always $x = 0$ and $y = 0$, then we shall, following Morita and Tachikawa [9], say that M and M^* form an *orthogonal pair* with respect to Q (and relative to the given product).[4] Now we have, as a generalization of [9, Theorem 1.1], the following fundamental

PROPOSITION 2. *Let Q be a quasi-Frobenius two-sided A-A^*-module. Suppose that a left A-module M and a right A^*-module M^* form an orthogonal pair with respect to Q, and suppose further that either M or M^* satisfies both the maximum and the minimum conditions for A- or A^*-submodules respectively. Then the other one also satisfies the same conditions, and moreover M and M^* may be regarded, in the natural manner, as the left-dual and the right-dual modules of M^* and M respectively.*

[4] In [9], the notion was defined in the special case where $A = A^*$. In this connection, the concept of dual modules was also introduced in this case, by the name of character modules.

The proof is virtually the same as that in the above cited theorem in [9], but we shall state it here for completeness. We may assume, without loss of generality, that M satisfies both the maximum and the minimum conditions, or equivalently, M has a composition series for A-submodules, say

$$M = M_0 \supset M_1 \supset \cdots \supset M_{s-1} \supset M_s = 0.$$

Consider then the following series of A^*-submodules of M^*:

$$M^* = r(M_s) \supseteq r(M_{s-1}) \supseteq \cdots \supseteq r(M_1) \supseteq r(M_0) = 0,$$

where $r(M_i)$ denotes, for each i, the right annihilator of M_i in M^*. The right multiplication of each element of $r(M_i)$ clearly induces in $M_{i-1} - M_i$ an A-homomorphism into Q and moreover elements of $r(M_{i-1})$ and only these induce the zero-mapping. Therefore $r(M_i) - r(M_{i-1})$ may be regarded as an A^*-submodule of the right-dual module of $M_{i-1} - M_i$. But since $M_{i-1} - M_i$ is irreducible and since Q is quasi-Frobenius, the right-dual of $M_{i-1} - M_i$ is either irreducible or zero, so that $r(M_i) - r(M_{i-1})$ is irreducible unless $r(M_i) = r(M_{i-1})$. It turns out from this that M^* has a composition series whose length $[M^*]_r$ does not exceed the composition length $s = [M]_l$ of M. Then we should, by symmetry, conclude that $[M]_l = [M^*]_r$. Now, M^* may be looked upon, in the natural manner, as an A^*-submodule of the right-dual module $\mathrm{Hom}_A(M, Q)$ of M. Therefore, M and $\mathrm{Hom}_A(M, Q)$ form also an orthogonal pair, and the later posesses a composition series whose length is equal to $[M]_l = [M^*]_r$. This implies that $M^* = \mathrm{Hom}_A(M, Q)$. Similarly, we have $M = \mathrm{Hom}_{A^*}(M^*, Q)$.

COROLLARY. *Under the same assumption as in Proposition 2, A-submodules L of M and A^*-submodules R of M^* correspond one-to-one by the following relations:*

$$r(L) = R, \qquad L = l(R),$$

where $r(\ )$ and $l(\ )$ mean the right and the left annihilators in M^* and M respectively; and, in this case, L and $M^* - R$, $M - L$ and R may respectively be regarded as dual modules of each other.

Proof. Let L be an A-submodule of M. Then L and $M^* - r(L)$ form naturally an orthogonal pair, so that L may be regarded as the left-dual module of the latter. But since $r(L)$ is also the right annihilator of $l(r(L))$, $l(r(L))$ is the left-dual module of $M^* - r(L)$ too. This implies, because of $L \subseteq l(r(L))$, that $L = l(r(L))$. Similarly, we have $R = r(l(R))$ for any A^*-submodule R of M^*.

PROPOSITION 3. *Let Q be a faithful two-sided A-A^*-module. In order that Q be quasi-Frobenius it is necessary and sufficient that the A-socle of Q contain the A^*-socle of Q and every A-homomorphism of any finitely generated completely reducible A-submodule of Q into Q can be given by the right multiplication of an element of A^*.*

Proof. Suppose that Q is quasi-Frobenius. That the A-socle contains (and in fact coincides with) the A^*-socle follows from Theorem 1. Let now L be a finitely generated completely reducible A-submodule of Q. Then L satisfies both the maximum and the minimum conditions. On the other hand, the left A-module L and the right A^*-module $A^* - r(L)$, $r(L)$ being the right annihilator of L in A^*, form an orthonal pair. Hence the latter may, by Proposition 2, be regarded as the right-dual module of L, or what is the same thing, every A-homomorphism of L into Q can be obtained by right-multiplying an element of A^*.

To prove the sufficiency, consider first a maximal left ideal $\mathfrak{l}$ of A such that $r_Q(\mathfrak{l}) \neq 0$. Take two non-zero elements u, v from $r_Q(\mathfrak{l})$. Then, by associating au, $a \in A$, with av, we have an A-isomorphism of Au onto Av, both of which are irreducible A-submodules isomorphic to $A - \mathfrak{l}$. The isomorphism may, therefore, be given by the right multiplication of an element a^* of A^*: $ua^* = v$, and this shows that $l_Q(\mathfrak{l})$ is A^*-irreducible. Consider next a maximal right ideal $\mathfrak{r}$ of A^* such that $r_Q(\mathfrak{r}) \neq 0$. Then $l_Q(\mathfrak{r})$ is contained in the (A^*- whence) A-socle and so it is a completely reducible A-submodule. Suppose that $l_Q(\mathfrak{r})$ were not irreducible. Then it would contain two distinct irreducible A-submodules $\mathfrak{m}$ and $\mathfrak{m}'$. Now the projection mappings of the direct sum $\mathfrak{m} \oplus \mathfrak{m}'$ onto $\mathfrak{m}$ and $\mathfrak{m}'$ can be given by the right multiplication of elements e and e' of A^* respectively. Consider the right annihilator $r(\mathfrak{m})$ of $\mathfrak{m}$ in A^*, which evidently contains $\mathfrak{r}$. Then e is not in $r(\mathfrak{m})$ but e' is in $r(\mathfrak{m})$. Hence we have $r(\mathfrak{m}) = \mathfrak{r}$ (because $\mathfrak{r}$ is maximal), so that e' is in $\mathfrak{r}$. But since $\mathfrak{m}' \subseteq l_Q(\mathfrak{r})$, it follows necessarily that $\mathfrak{m}'e' = 0$, and this is a contradiction. Thus it is proved that Q is quasi-Frobenius.

As an immediate specialization of Proposition 3, we have easily

THEOREM 2. *Let Q be a faithful two-sided A-A^*-module, and suppose that Q is completely reducible with respect to A. Then Q is quasi-Frobenius if and only if A^* is a dense subring*[5] *of the A-endomorphism ring of Q.*[6]

[5] Generally, a subring D of the endomorphism ring of an A-module Q is said to be dense if for any given finite number of elements $u_1, u_2, \cdots, u_n$ of Q and any given endomorphism f of Q, there exists an element d in D such that $u_1 d = u_1 f, u_2 d = u_2 f, \cdots, u_n d = u_n f$. Cf. [7].

[6] In view of this and Theorem 1, we can immediately deduce that if a faithful two-

Next, we proceed to the following

THEOREM 3. *Let Q be a faithful two-sided A-A^*-module, and suppose that Q is weakly distinguished with respect to A and every A-homomorphism of any finitely generated A-submodule of Q into the A-socle of Q can be given by the right multiplication of an element of A^*. Then Q is quasi-Frobenius.*

Proof. Consider a non-zero element u of Q, and denote by $l(u)$ the left annihilator of u in A. Associating the coset of any $a \in A$ modulo $l(u)$ with the element au, $A - l(u)$ is mapped isomorphically onto the cyclic A-submodule Au of Q. Since $l(u) \neq A$, there exists a maximal left ideal $\mathfrak{l}$ of A containing $l(u)$. Then evidently $Au - \mathfrak{l}u \cong A - \mathfrak{l}$, and therefore Q must, since it is weakly distinguished, contain an irreducible A-submodule isomorphic to $A - \mathfrak{l}$, that is, $r_Q(\mathfrak{l}) \neq 0$. Take any non-zero element v from $r_Q(\mathfrak{l})$. Then the mapping $au \rightarrow av$, $a \in A$, is obviously an A-homomorphism of Au onto the irreducible A-submodule Av, and hence there exists an element a^* of A^* such that $ua^* = v$. Thus we have shown that $uA^* \supseteq r_Q(\mathfrak{l})$. This implies in particular that every irreducible A^*-submodule of Q is of the form $r_Q(\mathfrak{l})$ with a suitable maximal left ideal $\mathfrak{l}$ of A (and conversely, any non-zero annihilator of such form is A^*-irreducible). Consequently, the A^*-socle of Q is contained in (and in fact coincides with) the A-socle of Q. Our theorem now follows immediately from Proposition 3.

Now we have the following main theorem:

THEOREM 4. *Let Q be an injective and distinguished left A-module, and let A^* be a dense subring of the A-endomorphism ring of Q. Then Q is quasi-Frobenius, when regarded as a two-sided A-A^*-module.*

Proof. By virtue of Theorem 3, we have only to prove that Q is faithful with respect to A. Let c be any non-zero element of A. Then the left annihilator $l(c)$ of c in A is a proper left ideal, and therefore $r_Q(l(c)) \neq 0$. Take now a non-zero element u from $r_Q(l(c))$. Then the mapping $ac \rightarrow au$,

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sided A-A^*-module Q is completely reducible with respect to A and if A^* is dense in the A-endomorphism ring of Q, then Q is also completely reducible with respect to A^* and A is dense in the A^*-endomorphism ring of Q, and conversely; moreover, in this case, homogeneous A-components and A^*-components of Q coincide. This fact, however, remains true even when A and A^* do not possess identity elements, as can easily be seen from the later remark at the end of this section, and therefore we have obtained Jacobson [7, Theorems VI. 1. 1 and VI. 2. 1]. Indeed, our proof of Proposition 3 may be seen, partly, as a modification of the proof of the former theorem.

$a \in A$, is an A-homomorphism of Ac into Q. Hence there exists, due to the injectivity of Q, an element v in Q such that $cv = u$, which shows that $cQ \neq 0$.

We shall next show that, under certain chain conditions, the converse of Theorems 3 and 4 is also true:

THEOREM 5. *Let Q be a quasi-Frobenius two-sided A-A^*-module. Then A satisfies the left minimum condition if and only if Q satisfies both the maximum and the minimum conditions for A^*-submodules. And, if this is the case,* (1) *A coincides with the A^*-endomorphism ring of Q,* (2) *Q is both injective and distinguished with respect to A,* (3) *A^* is a dense subring of the A-endomorphism ring of Q,*[7] (4) *every A^*-homomorphism of any A^*-submodule of Q into Q can be extended to an A^*-endomorphism of Q,* (5) *Q is weakly distinguished with respect to A^*.*

Proof. Since Q is faithful with respect to A, the left A-module A and the right A^*-module Q form an orthogonal pair (with respect to Q). If we apply Proposition 2 to this orthogonal pair, we get immediately the first assertion and (1). Apply next, assuming the chain conditions, the Corollary of Proposition 2 to the same orthogonal pair. Then, firstly, we know that, for every left ideal $\mathfrak{l}$ of A, $l_A(r_Q(\mathfrak{l})) = \mathfrak{l}$ holds and moreover $Q - r_Q(\mathfrak{l})$ is regarded as the right-dual module of $\mathfrak{l}$. The former fact implies that $r_Q(\mathfrak{l}) \neq 0$ whenever $\mathfrak{l} \neq A$, that is, Q is distinguished, while the latter fact means that every A-homomorphism of any $\mathfrak{l}$ into Q can be obtained by the right multiplication of an element of Q, that is, Q is A-injective. Secondly, we can see that, for any A^*-submodule R of Q, the factor module $A - l(R)$ modulo the left annihilator $l(R)$ of R in A may be regarded as the left-dual module of R, which means, in view of (1), nothing but (4). Let now L be a finitely generated A-submodule of Q, and denote by $r(L)$ the right annihilator of L in A^*. Then L satisfies both the maximum and the minimum conditions for A-submodules, and L and $A^* - r(L)$ form an orthonal pair. Hence, Proposition 2 can again be applied to conclude that every A-homomorphism of L into Q may be given by the right multiplication of an element of A^*, and this implies, in particular, (3). To show finally that Q is weakly A^*-distinguished, consider two A^*-submodules R and R' of Q such that $R \supset R'$ and $R - R'$ is A^*-irreducible. Then it follows again from the Corollary of Proposition 2 that $l(R) \neq l(R')$, and so we can find an element a in $l(R')$ which is not in $l(R)$. The left multiplication of a evidently maps

[7] This fact may be regarded as an extension of the density theorem for irreducible modules (cf. [7, p. 31]).

$R-R'$ isomorphically onto an irreducible A^*-submodule of Q. This completes the proof of our theorem.

By combining Theorem 4 with Theorem 5 (and by symmetry), we have the following special case:

THEOREM 6. *Let A and A^* be two rings, and let Q be a two-sided A-A^*-module. Then the following conditions are equivalent:*

(i) *A satisfies the left minimum condition, Q is injective, distinguished and finitely generated with respect to A, and A^* coincides with the A-endomorphism ring of Q.*[8]

(ii) *A^* satisfies the right minimum condition, Q is injective, distinguished and finitely generated with respect to A^*, and A coincides with the A^*-endomorphism ring of Q.*

(iii) *Q is quasi-Frobenius, and A and A^* satisfy, respectively, the left and the right minimum conditions.*

(iv) *Q is quasi-Frobenius, A satisfies the left minimum condition, and Q is finitely generated with respect to A.*

(v) *Q is quasi-Frobenius, A^* satisfies the right minimum condition, and Q is finitely generated with respect to A^*.*

We now turn to the general case. Let Q be a quasi-Frobenius two-sided A-A^*-module. Let M be an irreducible left A-module such that Q contains an isomorphic image of it, or equivalently, the right-dual module M^* of M is irreducible. Then M may be regarded as the left-dual module of M^* and moreover the homogeneous component of the socle of Q belonging to M coincides with the homogeneous component belonging to M^* (Theorem 1). Let Δ be the endomorphism division ring of M. Then [7, Theorem V. 7. 1], together with Proposition 3, implies that there is a lattice isomorphism between the lattice of A^*-submodules of the common homogeneous component and the lattice of Δ-submodules of M, and, in particular, the A^*-dimension of the homogeneous component coincides with the Δ-dimension of M. Thus we have

PROPOSITION 4. *Under the same assumptions as in Theorem 1, the A-*

[8] Here, the injectivity and the distinguishedness for Q may, by virtue of Theorem 3, be replaced respectively by the weaker conditions that every A-homomorphism of any A-submodule of Q into Q can be extended to an A-endomorphism of Q and that Q is weakly distinguished with respect to A.

and the A^*-dimensions of the homogeneous component of the socle belonging to M, or to M^*, coincide with the capacites of M^* and M respectively.

Now, we call a quasi-Frobenius Q **Frobenius** if, for any irreducible left A-module M such that Q contains an isomorphic image of it, the capacity of M coincides with that of the right-dual M^* of M, or what defines the same thing, if the A- and the A^*-dimensions of each homogeneous component of the socle of Q coincide.

The following proposition is not only for quasi-Frobenius modules but also for Frobenius modules and may be verified quite easily:

PROPOSITION 5. *Let Q be a quasi-Frobenius (or Frobenius) two-sided A-A^*-module and let Q_0 be an A-A^*-submodule of Q. Then $\mathfrak{z}=l_A(Q_0)$ and $\mathfrak{z}^*=r_{A^*}(Q_0)$ are two-sided ideals of A and A^* respectively, and Q_0 is quasi-Frobenius (or Frobenius) when regarded as a two-sided $A/\mathfrak{z}$—$A^*/\mathfrak{z}^*$-module.*

Assume again the left and the right minimum conditions for A and A^* respectively. Then left ideals $\mathfrak{l}$ of A and A^*-modules R of (the quasi-Frobenius module) Q correspond one-to-one by the annihilator relations (Corollary of Proposition 2), and it is evident that $\mathfrak{l}$ is a two-sided ideal if and only if the corresponding R is an A-A^*-submodule; the similar is also the case for right ideals of A^* and A-submodules of Q. This, together with Proposition 5, yields

THEOREM 7. *Let A and A^* satisfy the left and the right minimum conditions respectively, and let Q be a quasi-Frobenius (or Frobenius) two-sided A-A^*-module. Then there is a one-to-one correspondence between two-sided ideals $\mathfrak{z}$ of A, A-A^*-submodules Q_0 of Q, and two-sided ideals $\mathfrak{z}^*$ of A^* by the annihilator relations:*

$$r_Q(\mathfrak{z})=Q_0, \qquad \mathfrak{z}=l_A(Q_0),$$
$$r_{A^*}(Q_0)=\mathfrak{z}^*, \qquad Q_0=l_Q(\mathfrak{z}^*);$$

and, in this case, Q_0 is quasi-Frobenius (or Frobenius) when regarded as a two-sided $A/\mathfrak{z}$-$A^*/\mathfrak{z}^*$-module.

Example. Let A be a ring without non-zero nilpotent ideals. Then it is quasi-Frobenius when regarded as a two-sided A-module. To prove this, consider an irreducible left ideal $\mathfrak{l}$ of A. Then it is generated by an idempotent element e ([7, Proposition III. 9. 1]): $\mathfrak{l}=Ae$. One can now easily see that the right-dual module of $\mathfrak{l}$ with respect to A is isomorphic to eA, whereas the latter is an irreducible right ideal of A by virtue of [7, Corollary of Proposition IV. 3. 1]. Similarly, the left-dual module of every irreducible

right ideal of A is irreducible too, and this shows our assertion. It should be noted, in view of this, that [7, Theorem IV. 3. 1] may be interpreted as a special case of our Theorem 1.

Suppose next that A is a ring whose left socle S is faithful. Then A is semi-simple, and, in particular, it has no non-zero nilpotent ideals. Hence, it follows (from either of the above mentioned theorems) that S coincides with the right socle of A. Let $\mathfrak{z}$ be a non-zero two-sided ideal of A. Then $\mathfrak{z}S \neq 0$, and so there is an irreducible left ideal $\mathfrak{l}$ such that $\mathfrak{z}\mathfrak{l} \neq 0$, whence $\mathfrak{z}\mathfrak{l} = \mathfrak{l}$. But this implies $S\mathfrak{z} \neq 0$, because $S\mathfrak{z}\mathfrak{l} = S\mathfrak{l} \supseteq \mathfrak{l}^2 \neq 0$. Thus it is shown that S is a faithful right ideal of A, and we know from Proposition 5 that *if A is a ring having a faithful completely reducible left ideal, then (not only A but also) the common socle of A is quasi-Frobenius when regarded as a two-sided A-module.*

Remark. The notion of quasi-Frobenius two-sided A-A^*-modules Q may be transferred to the case when A and A^* do not necessarily have identity elements but Q satisfies $r_Q(A) = 0$ and $l_Q(A^*) = 0$, by taking the conditions i) and ii) as its definition, provided we restrict, in ii), both maximal left and maximal right ideals $\mathfrak{l}$ and $\mathfrak{r}$ to be modular (cf. [7, p. 5]). It is then almost evident that we may also replace ii) in this definition by the condition ii′). Furthermore, if we observe that the results stated in § 1, except those which are concerned with injectivity, remain valid for modular left ideals $\mathfrak{l}$ when A does not have an identity element but Q satisfies $r_Q(A) = 0$, we can examine, without difficulties, that all the propositions and the theorems in § 2, including the above example, still hold in our case under obvious additional assumptions, provided we require the existence of identity elements only for A in Theorem 4 as well as for A and A^* in (i) and (ii) of Theorem 6. For instance, we have to asume additionally that in Theorem 3, each element u of Q is always in Au, while in Theorem 5, A satisfies the left maximum condition. It is however to be noted that Theorem 2 remains true even without assuming that $r_Q(A) = 0$ and $l_Q(A^*) = 0$, because these conditions follow automatically from the complete reducibility for Q and the denseness of A^* respectively.

3. Duality theorems.

LEMMA 1. *Let Q be an injective left A-module and A^* its A-endomorphism ring. Let M be a left A-module and M^* its right-dual module with respect to Q. Then, for any $x \in M$ and $f \in M^*$, we have*

$$r_Q(l_A(x)) = xM^*, \qquad r(l(f)) = fA^*,$$

where $r(\ )$ *and* $l(\ )$ *in the second equality mean the right and the left annihilators in* M^* *and* M *respectively.*

Proof. Let u be an element in $r_Q(l_A(x))$. Then the mapping $ax \rightarrow au$, $a \in A$, is an A-homomorphism of Ax into Q, and it can, since Q is A-injective, be extended to an A-homomorphism $g(\in M^*)$ of M into Q: $xg = u$. This shows that $r_Q(l_A(x)) \subseteq$ whence $= xM^*$. Similarly, if h is an element in $r(l(f))$ the A-homomorphism $xf \rightarrow xh$, $x \in M$, can be extended to an A-endomorphism $a^*(\in A^*)$ of Q: $fa^* = h$, which shows $r(l(f)) = fA^*$.

PROPOSITION 6. *Under the same assumptions as in Lemma* 1, *suppose in addition that* Q *is* A*-distinguished. Then* $x = 0$ *is the only element of* M *such that* $xM^* = 0$. *More generally, we have* $l(r(L)) = L$ *for every* A*-submodule* L *of* M.

Proof. Let x be a non-zero element of M. Then $xM^* = r_Q(l_A(x))$ by Lemma 1, while the right side is non-zero because $l_A(x)$ is a proper left ideal of A and Q is distinguished. The second assertion may immediately be seen by applying this first one to the left A-module $M - L$ and its right-dual module $r(L)$.

We now get the following

THEOREM 8. *Let* A *and* A^* *be two rings satisfying the left and the right minimum conditions respectively and let* Q *be a quasi-Frobenius two-sided* A-A^**-module.* (*Or equivalently, let* A, A^* *and* Q *satisfy one of the equivalent conditions in Theorem* 6.) *Let* M *be a finitely generated left* A*-module and let* M^* *be its right-dual module with respect to* Q. *Then* M^* *is also finitely generated* (*with respect to* A^*) *and* M *coincides with the left-dual module of* M^*. *The same holds also for every finitely generated right* A^**-module.*

Proof. Since Q is both A-injective and A-distinguished, M and M^* form an orthogonal pair with respect to Q according to Proposition 6. On the other hand, the finitely generated left A-module M satisfies, because of the left minimum condition for A, both the maximum and the minimum conditions for A-submodules. Our theorem now follows immediately from Proposition 2.

We shall next prove the following converse of Theorem 8:

THEOREM 9. *Let* Q *be a two-sided* A-A^**-module. Suppose that, for each finitely generated left* A*-module* M, *the right-dual module* M^* *is also*

finitely generated and moreover M coincides with the left-dual module of M^*, and suppose that the same holds for each finitely generated right A^*-module. Then A and A^* satisfy the left and the right minimum conditions respectively and moreover Q is quasi-Frobenius.

Proof. Consider a finitely generated left A-module M and its right-dual module M^*. Let L be an A-submodule of M. Then the right annihilator $r(L)$ of L in M^* may be regarded as the right-dual module of $M-L$. Since $M-L$ is finitely generated, $r(L)$ is also finitely generated and moreover $l(r(L))=L$, according to our assumptions. Similarly, we know that, for each A^*-submodule R of M^*, the left annihilator $l(R)$ is finitely generated and moreover $r(l(R))=R$. These together show that A-submodules L of M and A^*-submodules R of M^* are all finitely generated and L and R correspond one-to-one by the annihilator relations. However, the former fact means, as is well-known, that M and M^* satisfy the maximum condition for A- and A^*-submodules respectively, and this, together with the latter fact, yields that M and M^* fulfill also the minimum condition. Now, we take in particular $M=A_s$ whence $M^*=Q$. Then it follows that A satisfies the left minimum condition, $l_A(Q)=0$, i.e., Q is A-faithful, and $r_Q(\mathfrak{l})$ is A^*-irreducible for every maximal left ideal $\mathfrak{l}$ of A. Furthermore, the similar must, by symmetry, be the case for A^* and the A-module Q, and this shows that Q is quasi-Frobenius.

Now suppose that M_1 and M_2 are two left (or right) A-modules. For any A-homomorphism $\phi\colon M_1\to M_2$ and any element $x\in M_1$, we denote by $x\phi$ (or ϕx) the image of x by ϕ. If further ψ is an A-homomorphism of M_2 into a third left (or right) A-module M_3, we shall denote by $\phi\circ\psi$ (or $\psi\circ\phi$) the composite mapping $x\to(x\phi)\psi$ (or $x\to\psi(\phi x)$). Let Q be a two-sided A-A^*-module, and let $M_1{}^*$ and $M_2{}^*$ be the right-dual modules of M_1 and M_2 respectively. Then we can associate with each ϕ an A^*-homomorphism $\phi^*\colon M_2\to M_1{}^*$ by setting $\phi^*g=\phi\circ g$, $g\in M_2{}^*$, i.e.,

$$x(\phi^*g)=(x\phi)g, \qquad x\in M_1, g\in M_2{}^*. \tag{*}$$

ϕ^* is called the *dual mapping* of ϕ with respect to Q, and it satisfies, with any A-homomorphism $\psi\colon M_2\to M_3$, $(\phi\circ\psi)^*=\phi^*\circ\psi^*$. Thus, the association $M^*\to M^*$, together with the mapping $\phi\to\phi^*$, defines a contravariant functor of one variable in the sense of [1]. Suppose now that A and A^* satisfy, respectively, the left and the right minimum conditions and Q is quasi-Frobenius. Suppose in addition that both M_1 and M_2 are finitely generated. Then M_1 and M_2 may, by Theorem 8, be looked upon as the left-dual modules of $M_1{}^*$ and $M_2{}^*$ respectively, and therefore the above equality (*) shows that

ϕ coincides with the dual mapping of ϕ^*. Thus, the mapping $\phi \to \phi^*$ gives an isomorphism between two groups $\mathrm{Hom}_A(M_1, M_2)$ and $\mathrm{Hom}_{A^*}(M_2^*, M_1^*)$. Moreover, it follows from Theorem 8 that the M^*'s exhaust, up to isomorphisms, all finitely generated right A^*-modules when M runs over all finitely generated left A-modules. We can however prove a complete converse of these situations:

THEOREM 10. *Let A and A^* be two rings. Suppose that we have a contravariant functor T (of one variable), defined only for finitely generated left A-modules M and taking finitely generated right A^*-modules as its values $T(M)$, such that the $T(M)$'s, up to isomorphisms, cover all finitely generated right A^*-modules and moreover T maps $\mathrm{Hom}_A(M_1, M_2)$ isomorphically onto $\mathrm{Hom}_{A^*}(T(M_2), T(M_1))$ for any finitely generated left A-modules M_1 and M_2. Then A and A^* satisfy the left and the right minimum conditions respectively, and there exists in fact a quasi-Frobenius two-sided A-A^*-module Q such that T is naturally equivalent*[9] *with the functor which is defined by associating every finitely generated left A-module M with its right-dual module M^* with respect to Q.*

Proof. There exists a finitely generated left A-module Q such that $T(Q)$ is isomorphic to the right A^*-module A^*. We may however assume, without loss of generality, that $T(Q) = A^*$. The left multiplication of an element a^* of A^* induces on A^* an A^*-endomorphism, and there must be a unique A-endomorphism of Q which is mapped by T onto it. If we identify this with a^*, Q can, since T is contravariant, be converted into a two-sided A-A^*-module. Consider a finitely generated left A-module M and its right-dual module $M^* = \mathrm{Hom}_A(M, Q)$. Then the mapping $f \to T(f)$, $f \in M^*$, is an isomorphism of M^* onto $\mathrm{Hom}_{A^*}(A^*, T(M))$), whereas the latter module may be identified naturally with $T(M)$.[10] Moreover, the above mapping is actually an A^*-isomorphism, because $T(fa^*) = (T(f \circ a^*) =) T(f) \circ T(a^*)$ should be identified with the element $T(f)a^*$ of $T(M)$. Now let M_1 and M_2 be two finitely generated left A-modules. Let ϕ be an A-homomorphism of M_1 into M_2 and $\phi^*\colon M_2^* \to M_1^*$ its dual mapping. Then $T(\phi^* g) = T(\phi \circ g) = T(\phi) \circ T(g)$ for any $g \in M_2^*$, and the last term is, when $T(g)$ is regarded as an element of $T(M_2)$, identified with $T(\phi)T(g)$, i.e., we have the commutativity of the following diagram:

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[9] Cf. [1, p. 20].

[10] That is, we identify each element of $T(M)$ with the multiplication effected by it in A^*; the element is conversely characterized as the image of the identity element of A^* by the identified A^*-homomorphism.

$$\begin{tikzcd} M_2^* \arrow[r, "T"] \arrow[d, "\phi^*"'] & T(M_2) \arrow[d, "T(\phi)"] \\ M_1^* \arrow[r, "T"] & T(M_1). \end{tikzcd}$$

Thus it is shown that T yields a natural equivalence between two functors $M \to M^*$ and T.

In order to prove the remaining part of our theorem, we may evidently assume that both functors coincide. Let M be a finitely generated left A-module, as above, and consider an A^*-homomorphism ξ of the right-dual module $T(M)$ of M into Q. Since Q is identified with the right-dual module $T(A)$ of A_s, there exists a unique A-homomorphism ϕ of A_s into M such that the dual mapping $T(\phi): f \to \phi \circ f$, $f \in M^*$, coincides with ξ. If x is an element of M which is identified with ϕ, then xf may also be identified with $\phi \circ f$ and so we have $xf = \xi f$ (for all $f \in M^*$). Thus it is shown that M coincides with the left-dual module of $T(M)$. Since moreover the $T(M)$'s range, up to isomorphisms, over all finitely generated right A^*-modules, it follows from Theorem 9 that A and A^* satisfy the left and the right minimum conditions respectively and Q is quasi-Frobenius.

Remark. Lemma 1 may be regarded as an extension of the first part of Ikeda and Nakayama [6, Theorem 1]. And by making use of Lemma 1, we can easily generalize the last part of this theorem in the following form: *under the same assumptions as in Lemma* 1, $r(L_1 \cap L_2) = r(L_1) + r(L_2)$ *for any A-submodules L_1 and L_2 of M, and $r(l(R)) = R$ for all finitely generated A^*-submodules R of M^*.* As an immediate consequence of this, we know that if M satisfies the minimum condition for A-submodules, then M^* satisfies the maximum condition for A^*-submodules, and moreover if M satisfies both the maximum and the minimum conditions for A-submodules, then so does M^* for A^*-submodules. In particular, if A satisfies the left minimum condition, then Q satisfies both the maximum and the minimum conditions for A^*-submodules, while if Q satisfies both the maximum and the minimum conditions for A-submodules, then A^* satisfies the right minimum condition.

4. Injective modules and quasi-Frobenius rings. Let A be a ring, and let M be a left A-module. An extension A-module M' of M is called an *essential extension* of M if $M'' = 0$ is the only A-submodule of M' such that $M'' \cap M = 0$. After showing that injective modules may be characterized as those modules which have no essential extensions other than themselves,

Eckmann and Schopf [2] proved the existence and the uniqueness (up to isomorphisms over M) of an injective essential extension $\hat{M}$ of any given (left A-module) M. Moreover, every injective extension of M contains such an $\hat{M}$, and therefore injective essential extensions are nothing but minimal injective extensions.

PROPOSITION 7. *An injective left A-module $Q(\neq 0)$ is directly indecomposable if and only if it is an essential extensions of every non-zero A-submodule.*

Proof. Suppose that Q is directly indecomposable. Let M be a non-zero A-submodule. Then Q contains an injective essential extension $\hat{M}$ of M. Since $\hat{M}$ is a direct summand of Q, we must have $Q=\hat{M}$. Suppose, conversely, Q is directly decomposable into two non-zero A-submodules, say, M and M': $Q=M\oplus M'$. Then necessarily $M\cap M'=0$, which shows that Q is not an essential extension of M.

COROLLARY. *Let Q be an injective left A-module containing an irreducible A-submodule M. Then the following conditions are equivalent:*

(i) *Q is directly indecomposable.*

(ii) *M is a smallest A-submodule*[11] *of Q.*

(iii) *Q is an essential extension of M.*

Proof. The implication (i)$\Rightarrow$(iii) is the special case of the preceding proposition, while the implications (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) hold evidently without the assumption of injectivity for Q.

PROPOSITION 8. *If A satisfies the left maximum condition, a direct sum of left A-modules is injective if and only if so is each direct summand.*[12]

Proof. The "only if" is well-known and is easy to see. So we have only to prove the "if" part. Suppose that a left A-module Q is a direct sum of injective A-submodules Q_μ: $Q=\sum_\mu \oplus Q_\mu$, and denote by ϵ_μ the projection mapping of Q onto Q_μ for each μ. Let $\mathfrak{l}$ be a left ideal of A, and suppose that there is given an A-homomorphism $\phi:\mathfrak{l}\to Q$. Then there exists, since the composite mapping $\phi\circ\epsilon_\mu$ is an A-homomorphism of $\mathfrak{l}$ into Q_μ, an element u_μ in Q_μ such that $(a\phi)\epsilon_\mu=(a(\phi\circ\epsilon_\mu)=)au_\mu$ for all $a\in\mathfrak{l}$; here,

[11] By a smallest A-submodule we mean a non-zero A-submodule which is contained in all non-zero A-submodules; it is of course the only minimal A-submodule, if it exists.

[12] Cf. [1, Exercise I. 8].

we may of course take $u_\mu = 0$ whenever $(\mathfrak{l}\phi)\epsilon_\mu = 0$. But since A satisfies the left maximum condition, $\mathfrak{l}$ and hence $\mathfrak{l}\phi$ is finitely generated, so that $(\mathfrak{l}\phi)\epsilon_\mu = 0$, or $u_\mu = 0$ except for only a finite number of μ. Now their sum $u = \sum_\mu u_\mu$ fulfills $au = a\phi$ for all $a \in \mathfrak{l}$, and this shows the injectivity of Q.

Suppose, from now on, that A satisfies the left minimum condition. Let N be the radical of A, and let $\bar{A}$ denote the (semi-simple) factor ring A/N, or the factor module $A - N$. Then $\bar{A}$ is a direct sum sum of orthogonal simple subrings $\bar{A}_1, \bar{A}_2, \cdots, \bar{A}_k$. Let $\bar{e}_\kappa$ be, for each κ, a primitive idempotent element in $\bar{A}_\kappa$. Then $\bar{e}_\kappa \bar{A} \bar{e}_\kappa$ is a division ring, and $\bar{A}_\kappa$ is (ring-)isomorphic with the total matric algebra over $\bar{e}_\kappa \bar{A} \bar{e}_\kappa$ of a finite degree $f(\kappa)$. In fact, $\bar{A}\bar{e}_\kappa$ is an irreducible left $\bar{A}$- or A-module whose endomorphism ring is $\bar{e}_\kappa \bar{A} \bar{e}_\kappa$ and whose capacity is $f(\kappa) : [\bar{A}\bar{e}_\kappa : \bar{e}_\kappa \bar{A} \bar{e}_\kappa] = f(\kappa)$, and $\bar{A}_\kappa$ is, as left A-module, isomorphic to the $f(\kappa)$-times direct sum of $\bar{A}\bar{e}_\kappa$: $\bar{A}_\kappa \cong (\bar{A}\bar{e}_\kappa)^{f(\kappa)}$. Moreover, the k modules $\bar{A}\bar{e}_1, \bar{A}\bar{e}_2, \cdots, \bar{A}\bar{e}_k$ exhaust, up to isomorphisms, all irreducible left A-modules. There exists, for each κ, an idempotent representative $e_\kappa (\in A)$ of the coset $\bar{e}_\kappa$. The k idempotent elements $e_1, e_2, \cdots, e_k$ are all primitive and non-isomorphic, and any primitive idempotent element of A is isomorphic to one of them. Furthermore, A is, as left A-module, isomorphic to the direct sum $\sum_{\kappa=1}^{k} \oplus (Ae_\kappa)^{f(\kappa)}$; the isomorphism naturally yields a decomposition of the identity element 1 of A into orthogonal primitive elements $e_{\kappa i}$, $\kappa = 1, 2, \cdots, k$, $i = 1, 2, \cdots, f(\kappa)$, such that $e_{\kappa i}$ is, for each κ, isomorphic to e_κ and $A = \sum_{\kappa=1}^{k} \oplus \sum_{i=1}^{f(\kappa)} \oplus Ae_{\kappa i}$ gives a direct decomposition of A into directly indecomposable left ideals.

Now, we denote by Q_κ the minimal injective extension of the irreducible left-A-module $\bar{A}\bar{e}_\kappa$. Then, according to the Corollary of Proposition 7, Q_κ is directly indecomposable and has $\bar{A}\bar{e}_\kappa$ as a smallest A-submodule, and conversely any directly indecomposable injective left A-module is isomorphic with some Q_κ. Consider a left A-module Q. Then the right annihilator $r_Q(N)$ of the radical N is, as is well-known, the socle of Q, and moreover Q is its essential extension. Suppose now $r_Q(N) \cong \sum_{\kappa=1}^{k} \oplus (\bar{A}\bar{e}_\kappa)^{g(\kappa)}$ with non-negative cardinal numbers $g(\kappa)$, and consider the direct sum $Q' = \sum_{\kappa=1}^{k} \oplus (Q_\kappa)^{g(\kappa)}$. Then Q' is injective by virtue of Proposition 8, and therefore the above isomorphism of $r_Q(N)$ onto $\sum_{\kappa=1}^{k} \oplus (\bar{A}\bar{e}_\kappa)^{g(\kappa)}$ can be extended to an isomorphism of Q into Q' ([2, 4. i. 2]). Since, however, $\sum_{\kappa=1}^{k} \oplus (\bar{A}\bar{e})_\kappa{}^{g(\kappa)}$ is the socle of Q',

Q' is necessarily an essential extension of the isomorphic image of Q. From this follows immediately the following generalization of Nagao and Nakayama [10, Theorem 2]:

THEOREM 11. *A left module of a ring A satisfying the left minimum condition is injective if and only if it is a (finite or infinite) direct sum of A-submodules each of which is isomorphic to some Q_κ, where Q_κ is the minimal injective extension of an irreducible left A-module $\bar{A}\bar{e}_\kappa$ which is the homomorphic image of a directly indecomposable left component Ae_κ of A.*

Let Q be an injective left A-module, i.e., $Q \cong \sum_{\kappa=1}^{k} \oplus (Q_\kappa)^{g(\kappa)}$, with uniquely determined multiplicities $g(\kappa)$. It is then evident that Q is distinguished if and only if $g(\kappa) \neq 0$ for all κ, and Q is, in this case, finitely generated if and only if all Q_κ are finitely generated and all $g(\kappa)$ are finite. Suppose now that this is the case, that is, Q is both distinguished and finitely generated. Let A^* be the A-endomorphism ring of Q. Then A^* satisfies the right minimum condition by Theorem 6, and there exist in A^* a complete system of non-isomorphic primitive idempotents $e_1^*, e_2^*, \cdots, e_k^*$ (just like $e_1, e_2, \cdots, e_k$ in A) such that $Qe_\kappa^* \cong Q_\kappa$ for each κ. On the other hand, consider an A^*-module $e_\kappa Q$, which is a direct summand of the right A^*-module Q. Since Q is also A^*-injective and A coincides with its A^*-endomorphism ring again by Theorem 6, $e_\kappa Q$ is necessarily both injective and directly indecomposable, and therefore it contains, according to the Corollary of Projosition 7, a smallest A^*-submodule. These facts show the necessity of the following theorem:

THEOREM 12. *Let A and A^* be two rings satisfying the left and the right minimum conditions respectively and let $e_1, e_2, \cdots, e_k$ be a complete system of non-isomorphic primitive idempotent elements in A. Let Q be a faithful two-sided A-A^*-module. In order that Q be quasi-Frobenius it is necessary and sufficient that A^* have exactly k non-isomorphic primitive idempotent elements $e_1^*, e_2^*, \cdots, e_k^*$ and they, if suitably ordered, satisfy the following conditions:*

i) *the left A-module Qe_κ^* contains, for each κ, a smallest A-submodule and this is isomorphic to $\bar{A}\bar{e}_\kappa$, the homomorphic image of Ae_κ modulo the radical of A.*

ii) *the right A^*-module $e_\kappa Q$ contains, for each κ, a smallest A^*-submodule.*

And, if this is the case, the smallest A^*-submodule of $e_\kappa Q$ is isomorphic to $\bar{e}_\kappa^* \bar{A}^*$, the homomorphic image of $e_\kappa^* A^*$ modulo the radical of A^*.

The sufficiency can, however, be proved in the similar way as in (the first half of) the proof of Nakayama [12, Theorem 6]. Namely, let $e_{\kappa i}^*$, $\kappa = 1, 2, \cdots, k$, $i = 1, 2, \cdots, g(\kappa)$, denote similar orthogonal primitive idempotent elements in A^* as $e_{\kappa i}$ in A: $e_{\kappa i}^* \cong e_\kappa^*$, $1 = \sum_{\kappa=1}^{k} \sum_{i=1}^{g(\kappa)} e_{\kappa i}^*$. In view of the fact that $r_Q(N)$ is an A-A^*-submodule of Q, we have then the direct decomposition $r_Q(N) = \sum_{\kappa=1}^{k} \oplus \sum_{i=1}^{g(\kappa)} \oplus\, r_Q(N) e_{\kappa i}^*$, where $r_Q(N) e_{\kappa i}^* \cong r_Q(N) e_\kappa^*$ (as left A-modules). But since $r_Q(N)$ is the A-socle of Q, $r_Q(N) e_\kappa^* = r_Q(N) \cap Q e_\kappa^*$ is by virtue of the assumption i), necessarily the smallest A-submodule of $Q e_\kappa^*$ and is isomorphic to $\bar{A}\bar{e}_\kappa$. This implies that each $\sum_{i=1}^{g(\kappa)} \oplus\, r_Q(N) e_{\kappa i}^*$ is a homogeneous (A-)component of $r_Q(N)$, and hence is an A-A^*-submodule. Moreover, it is a minimal A-A^*-submodule. For, if u is any non-zero element in it then $u e_{\kappa p}^* \neq 0$ for some p; but $A u e_{\kappa p}^* = r_Q(N) e_{\kappa p}^*$, because of the irreducibility of the right side, and therefore

$$A u A^* \supseteq A u e_{\kappa p}^* A^* = r_Q(N) e_{\kappa p}^* A^* \supseteq \sum_{i=1}^{g(\kappa)} \oplus\, r_Q(N) e_{\kappa i}^*.$$

Let now N^* be the radical of A^*. Then every minimal two-sided A-A^*-module is, since N^* is nilpotent, annihilated by N^*, and, in particular, we have $\sum_{i=1}^{g(\kappa)} \oplus\, r_Q(N) e_{\kappa i}^* \subseteq l_Q(N^*)$. Since this is true for every κ, it follows that $r_Q(N) \subseteq l_Q(N^*)$. Now, $l_Q(N^*)$ is the A^*-socle of Q and hence $e_\kappa l_Q(N^*) = l_Q(N^*) \cap e_\kappa Q$ is, by the assumption ii), the smallest A^*-submodule of $e_\kappa Q$. But since $e_\kappa r_Q(N) e_\kappa^* \subseteq e_\kappa l_Q(N^*) e_\kappa^*$ and $e_\kappa r_Q(N) e_\kappa^* \neq 0$ (because $r_Q(N) e_\kappa^* \cong \bar{A}\bar{e}_\kappa$), we have $e_\kappa l_Q(N^*) e_\kappa^* \neq 0$, whence $e_\kappa l_Q(N^*) \cong \bar{e}_\kappa^* \bar{A}^*$. From this and by symmetry, we can conclude that $r_Q(N)$ ($\supseteq$ whence) $= l_Q(N^*)$; observe that we have derived the above relation $r_Q(N) \subseteq l_Q(N^*)$ from the condition i) only.

Consider now any irreducible left A-module $\bar{A}\bar{e}_\kappa$. It is isomorphic to the factor module $A - \mathfrak{l}$ modulo the maximal left ideal $\mathfrak{l} = A(1 - e_\kappa) + N$. Hence the right-dual module of $\bar{A}\bar{e}_\kappa$ with respect to Q is isomorphic to $r_Q(\mathfrak{l}) = r_Q(1 - e_\kappa) \cap r_Q(N) = e_\kappa Q \cap r_Q(N) = e_\kappa r_Q(N) = e_\kappa l_Q(N^*)$, which is A^*-irreducible as we have seen just above. Similarly, we may show that the left-dual module of every irreducible right A^*-module is A-irreducible too, and thus Q is quasi-Frobenius.

Now, Nakayama [12] called A, satisfying both the left and the right minimum conditions, to be a *quasi-Frobenius ring* if there exists a permutation $(\pi(1), \pi(2), \cdots, \pi(k))$ of $(1, 2, \cdots, k)$ such that for each κ,

i) $Ae_{\pi(\kappa)}$ contains a smallest left subideal and this is isomorphic to $\bar{A}\bar{e}_\kappa$,

ii) $e_\kappa A$ contains a smallest right subideal.

So, Theorem 12, combined with Theorem 6, yields the following

THEOREM 13. *A ring A satisfying the left minimum condition is a quasi-Frobenius ring if and only if it is quasi-Frobenius when regarded as a two-sided A-module.*

Owing to the preceding theorem, it follows from the Corollary of Proposition 2 in particular that *if A is a quasi-Frobenius ring, then $l(r(\mathfrak{l}))=\mathfrak{l}$ and $r(l(\mathfrak{r}))=\mathfrak{r}$ for every left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$ of A.* Consider now a maximal right ideal $\mathfrak{r}$ of A such that $l(\mathfrak{r})\neq 0$. Let $\mathfrak{l}$ be an irreducible left subideal of $l(\mathfrak{r})$. Then $r(\mathfrak{l})=\mathfrak{r}$, because $r(\mathfrak{l})$ is clearly a proper right ideal containing $\mathfrak{r}$. From this and by symmetry, we can easily conclude that *if conversely $l(r(\mathfrak{l}))=\mathfrak{l}$ and $r(l(\mathfrak{r}))=\mathfrak{r}$ for every irreducible left ideal $\mathfrak{l}$ and irreducible right ideal $\mathfrak{r}$ of A, satisfying the left minimum condition, then A is quasi-Frobenius.*[13] Thus we have derived [12, Theorem 6]. Next, let n be a natural number and consider the direct sum A^n of n copies of A. A^n may be regarded as a left as well as a right A-module in the usual fashion, and it is evident that if we define the product of twe vectors $(a_1,a_2,\cdots,a_n)$ and $(b_1,b_2,\cdots,b_n)$ to be $a_1b_1+a_2b_2+\cdots+a_nb_n$, the left A-module A^n and the right A-module A^n form an orthogonal pair with respect to A. Hence, we have, as a special case of the Corollary of Proposition 2, the following theorem of Hall [4]:[14] *If A is a quasi-Frobenius ring then $l(r(L))=L$ and $r(l(R))=R$ for every left A-submodule L and right A-submodule R of A^n.* On the other hand, Theorem 6 shows that a necessary and sufficient condition for A to be a quasi-Frobenius ring is that A be both injective and distinguished as left A-module. However, we may, for sufficient condition, omit the distinguishedness. To prove this, suppose that A is injective. Then each direct summand Ae_κ is also injective, so that it is, by (the Corollary of) Proposition 7, a minimal injective extension of its irreducible (in fact, smallest) left subideal $\mathfrak{l}_\kappa$. The k irreducible left ideals $\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k$ are mutually non-isomorphic, since so are $Ae_1,Ae_2,\cdots,Ae_k$, and therefore they must coincide, up to isomorphisms and up to arrangements, with $\bar{A}\bar{e}_1,\bar{A}\bar{e}_2,\cdots,\bar{A}\bar{e}_k$. We thus obtain the following theorem of Ikeda [5] (cf. also [6], Eilenberg and Nakayama [3]): *A ring satisfying the left*

[13] We may here restrict further $\mathfrak{l}$ and $\mathfrak{r}$ to be nilpotent, because every non-nilpotent irreducible one-sided ideal is generated by an idempotent element and so satisfies the above annihilator relations.

[14] In [9], this was derived in just the same way. Cf. also [12, Theorem 12].

minimum condition is quasi-Frobenius if and only if is is left self-injective.

Now, consider again an injective left A-module $Q = \sum_{\kappa=1}^{k} \oplus (Q_\kappa)^{g(\kappa)}$, and suppose that Q is both finitely generated and distinguished, or what is the same thing, all Q_κ are finitely generated and all $g(\kappa)$ are non-zero and finite. Then the endomorphism ring A^* of Q satisfies the right minimum condition, and Q is quasi-Frobenius as two-sided A-A^*-module (Theorem 6); moreover, it is easy to see that, on retaining the notations in Theorem 12, the capacity of each irreducible right A^*-module $\bar{e}_\kappa{}^*\bar{A}^*$ is $g(\kappa)$. From this (and Theorem 6), we get immediately the following

THEOREM 14. *Let Q be a two-sided A-A^*-module, where A satsifies the left minimum condition. Then in order that Q be Frobenius and that A^* satisfy the right minimum condition it is necessary and sufficient that Q be, as left A-module, both isomorphic to $\sum_{\kappa=1}^{k} \oplus (Q_\kappa)^{f(\kappa)}$, the minimal injective extension of $\bar{A} = A - N$, and finitely generated and that moreover A^* be the A-endomorphism ring of Q.*

Nakayama [12] called A a *Frobenius ring* if it is quasi-Frobenius and moreover $f(\kappa) = f(\pi(\kappa))$ for every $\kappa = 1, 2, \cdots, k$. So we have, in particular,

THEOREM 15. *A ring A satisfying the left minimum condition is a Frobenius ring if and only if it is Frobenius when regarded as a two-sided A-module, that is, it is, as left A-module, isomorphic to the minimal injective extension of $\bar{A} = A - N$.*

As an application of Theorems 14 and 15, we prove the following

THEOREM 16.[15] *Let A be a Frobenius ring, and let $\mathfrak{z}$ be a two-sided ideal of A. Then the factor ring $A/\mathfrak{z}$ is Frobenius if and only if the right annihilator $r(\mathfrak{z})$ of $\mathfrak{z}$ is both left and right principal:*[16] *$r(\mathfrak{z}) = Ac = cA$. And, in this case, $A/\mathfrak{z}$ is isomorphic with $A/\mathfrak{z}^*$, where $\mathfrak{z}^*$ is the double right annihilator of $\mathfrak{z}$: $\mathfrak{z}^* = r(r(\mathfrak{z}))$.*

Proof. By Theorem 7, $r(\mathfrak{z})$ is Frobenius when regarded as a two-sided

[15] This is a generalization of [11, Theorem 9] and is also a modification of [12, Theorem 15]. There is a little discrepancy between our theorem and the latter theorem; in fact, the "only if" part of the latter is contained in that part of ours (and its left analogy), but the similar is not the case for the "if" parts.

[16] Generally, in a ring satisfying both the left and the right minimum conditions, a two-sided ideal which is both left and right principal is generated by a common element. See Nakayama [13, Lemma 1].

$A/\mathfrak{z}$-$A/\mathfrak{z}^*$-module. Hence, in order that $A/\mathfrak{z}$ be Frobenius it is necessary and sufficient, by virtue of Theorems 14 and 15, that $A/\mathfrak{z}$ be, as left ($A/\mathfrak{z}$- or) A-module, isomorphic to $r(\mathfrak{z})$, or what is the same thing, that there exist an element c in $r(\mathfrak{z})$ such that $Ac = r(\mathfrak{z})$ and $l(c) = \mathfrak{z}$; but the latter equality means $cA = r(\mathfrak{z})$, because $r(l(c)) = r(l(cA)) = cA$ and $l(r(\mathfrak{z})) = \mathfrak{z}$. The last assertion is now an immediate consequence of the fact that $A/\mathfrak{z}$ and $A/\mathfrak{z}^*$ are, in this case, the endomorphism rings of the left A-modules $A/\mathfrak{z}$ and $r(\mathfrak{z})$ respectively; in fact, we have an isomorphism between these two rings by associating $a \in A$ with such $a^* \in A^*$ that $ac = ca^*$.

Finally, we would like to refer to the existence of finitely generated injective modules. Let r be the index of nilpotency of N. Consider a factor module $N^i - N^{i+1}$, $1 \leqq i < r$, and any left A-module M. Then the module $\mathrm{Hom}_A(N^i - N^{i+1}, M)$ consisting of all A-homomorphisms h of $N^i - N^{i+1}$ into M can be converted into a left A-module by setting ah, $a \in A$, to be the mapping $x \to (xa)h$, $x \in N^i - N^{i+1}$ (cf. [1, II. 3.]). Now, Rosenberg and Zelinsky proved the following theorem ([14, Theorem 1]): *The minimal injective extension $\hat{M}$ of M is finitely generated if and only if so is every left A-module* $\mathrm{Hom}_A(N^i - N^{i+1}, M)$, $i = 1, 2, \cdots, r-1$. Moreover, they gave, by making use of this theorem, an example for a ring A (satisfying the left minimum condition but) having no finitely generated injective left-module ($\neq 0$). We shall, however, need later only the following special case of the above theorem:

PROPOSITION 9. *Let A be a commutative ring satisfying the minimum condition for ideals, and let M be a finitely generated A-module. Then the minimal injective extension of M is also finitely generated.*

For, $\mathrm{Hom}_A(N^i - N^{i+1}, M)$ may be, in this case, interpreted as the dual module of $N^i - N^{i+1}$ with respect to M (when M is regarded as a two-sided A-module in the natural manner), and therefore is finitely generated, as can easily be seen from the finite generatedness for both modules $N^i - N^{i+1}$ and M.

Addendum. It is perhaps of some interest, in connection with Theorem 7, to add the following theorem, which holds for an arbitrary ring A:

THEOREM 17. *Let M be a left A-module and $Q = \hat{M}$ its minimal injective extension. Let $\mathfrak{z}$ be a two-sided ideal of A. Then $r_Q(\mathfrak{z})$, when regarded as a left-module of $A/\mathfrak{z}$, is the minimal injective extension of the left $A/\mathfrak{z}$-module $r_M(\mathfrak{z})$.*

Proof. It is easy to see that $r_Q(\mathfrak{z})$ is $A/\mathfrak{z}$-injective. So it suffices to show that $r_Q(\mathfrak{z})$ is, as $A/\mathfrak{z}$-module, an essential extension of $r_M(\mathfrak{z})$. However,

this follows immediately from the fact that $M' \cap M = M' \cap r_Q(\mathfrak{z}) \cap M = M' \cap r_M(\mathfrak{z})$ for every A-submodule M' of $r_Q(\mathfrak{z})$.

COROLLARY. *Let $\mathfrak{z}$ be a two-sided ideal of A, and let M be an injective left $A/\mathfrak{z}$-module. Looking upon M as an A-module, let Q be the minimal A-injective extension of M. Then we have $r_Q(\mathfrak{z}) = M$.*

5. The canonical module for an algebra. Changing letters, let Φ be a commutative ring satisfying the minimum condition (for ideals) and let $\mathfrak{n}$ be the radical of Φ. Denote by F the minimal injective extension of the (completely reducible) factor module $\Phi - \mathfrak{n}$, and let us call it the *canonical Φ-module.*

PROPOSITION 10. *The canonical Φ-module F is finitely generated, and Φ coincides with the Φ-endomorphism ring of F; in other words, F is a Frobenius Φ-module, when regarded as a two-sided Φ-module in the natural manner.*

Proof. The finite generatedness of F follows from Proposition 9. Let Φ^* be the Φ-endomorphism ring of F. Then, by Theorem 14, Φ^* satisfies the right minimum condition and F is a Frobenius two-sided Φ-Φ^*-module. Furthermore, since $r_F(\mathfrak{n}) = (l_F(\mathfrak{n}) =) l_F(\mathfrak{n}\Phi^*)$, it follows from Theorem 7 that $r_F(\mathfrak{n})$ is Frobenius as two-sided $\Phi/\mathfrak{n}$-$\Phi^*/\mathfrak{n}\Phi^*$-modlue. On the other hand, the Φ-socle $r_F(\mathfrak{n})$ of F is (isomorphic to) $\Phi - \mathfrak{n}$, so that $\Phi/\mathfrak{n}$ coincides with the Φ-endomorphism ring of $r_F(\mathfrak{n})$. This implies that $\Phi^*/\mathfrak{n}\Phi^* = \Phi/\mathfrak{n}$, i.e., $\Phi^* = \Phi + \mathfrak{n}\Phi^*$; but since $\mathfrak{n}$ is nilpotent, we can immediately deduce that $\Phi^* = \Phi$.

Combining Proposition 10 again with Theorem 7, we have

COROLLARY. *Ideals $\mathfrak{a}$ of Φ and Φ-submodules F_0 of the canonical Φ-module F correspond one-to-one by the annihilator relations, and in this case, F_0 is regarded as the canonical $\Phi/\mathfrak{a}$-module in the natural way.*

Let now M be a finitely generated Φ-module. According to Theorem 8, the dual module $\tilde{M} = \mathrm{Hom}_\Phi(M, F)$ of M with respect to F is also a finitely generated Φ-module, and M coincides with the dual module of $\tilde{M}$; furthermore, Φ-submodules L of M and R of $\tilde{M}$ correspond one-to-one by the annihilator relations, so that L and $\tilde{M} - R$ are dual modules of each other.

We shall, from now on, assume that A is an *algebra* over Φ in the sense that A is a ring and at the same time a finitely generated Φ-module such that $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for $\alpha \in \Phi$, $a, b \in A$.[17] Then A satisfies evi-

[17] It should be noted that here the notion of algebras is free of such a condition as that they have linearly independent bases, or more generally that they are projective,

dently the left as well as the right minimum condition. So we may use notations N, $\bar{A}=A/N(=A-N)$, k, $f(\kappa)$, e_κ, etc. in the same meanings as in the preceding section. Let now M be a finitely generated left (or right) A-module. Looking upon M as a finitely generated Φ-module in the natural way, we consider its dual module $\tilde{M}$ (with respect to F). We may however convert $\tilde{M}$ into a right (or left) A-module by setting χa (or $a\chi$), $\chi \in \tilde{M}$, $a \in A$, to be the mapping $x \to \chi(ax)$ (or $x \to \chi(xa)$), $x \in M$. The A-module $\tilde{M}$ is obviously finitely generated, and will be called the *Φ-dual module* of (the A-module) M. It is then easy to see that M may be regarded as the Φ-dual module of $\tilde{M}$ in the natural manner and there exists a one-to-one correspondence between A-submodules L of M and R of $\tilde{M}$ by the annihilator relations, so that L and $\tilde{M}-R$, $M-L$ and R are Φ-dual modules of each other[18]; the corresponding L and R we shall call the *Φ-annihilators* of R and L respectively. From this it follows in particular that $\tilde{M}$ is irreducible if and only if so is M, and indeed we have

LEMMA 2. *The Φ-dual module of $\bar{A}\bar{e}_\kappa$ is isomorphic to $\bar{e}_\kappa\bar{A}$, for each κ.*

For, if χ is an element of the Φ-dual module $\tilde{M}$ of $M=\bar{A}\bar{e}_\kappa$ and if $\lambda \neq \kappa$, then χe_λ maps M onto $\chi(e_\lambda M)=\chi(\bar{e}_\lambda\bar{A}\bar{e}_\kappa)=0$, that is, $\tilde{M}e_\lambda=0$.

Let Q be a two-sided A-module (which is element-wise commutative with Φ). Then it is clear that the Φ-dual module $\tilde{Q}$ of Q is (not only a right and a left but also) a two-sided A-module too. We now call Q a *canonical two-sided A-module* if $\tilde{Q}$ is isomorphic to (the two-sided A-module) A itself, or equivalently, if Q is isomorphic to the Φ-dual module $\tilde{A}$ of A.

THEOREM 18. *In order that a two-sided A-module Q be canonical it is necessary and sufficient that Q be faithful and have a Φ-homomorphism μ into F such that $\mu(au)=\mu(ua)$ for $a \in A$ and $u \in Q$ and that $\mu(L)\neq 0$ for every non-zero left A-submodule L of Q.*

Proof. That $\tilde{Q}\cong A$ means the existence of an element μ of $\tilde{Q}$ satisfying the following three conditions: (1) $a\mu=\mu a$, $a \in A$, (2) $\mu A=\tilde{Q}$, (3) $\mu a=0$, $a \in A$, implies $a=0$. However, the second condition (2) is equivalent to saying that the Φ-annihilator of μA is 0, that is, $\mu(Au)=0$, $u \in Q$, implies $u=0$, while the third condition (3) means nothing but that $\mu(aQ)=0$, $a \in A$,

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with respect to their base rings, which was asumed throughout in both the papers [3] and Kasch [8] (although we impose stronger restrictions on base rings in our case).

[18] In the special case where Φ is a field, F coincides necessarily with Φ itself, and therefore the concept of Φ-dual modules accords with that of dual representation spaces; indeed, the above relationship between Φ-dual modules turns, in this case, to the known relationship mentioned at the beginning of our introduction.

implies $a = 0$, which, under both the assumptions (1) and (2), is evidently equivalent to the faithfulness of Q.

LEMMA 3. *For any left (or right) ideal of A, its right (or left) annihilator in $\bar{A}$ coincides with its Φ-annihilator.*

Proof. Suppose that χ is an element of the right annihilator in $\bar{A}$ of the given left ideal $\mathfrak{l}$. Then $\chi(a) = (a\chi)(1) = 0$ for all $a \in \mathfrak{l}$. Assume conversely that χ is in the Φ-annihilator of $\mathfrak{l}$. Then, for any $a \in \mathfrak{l}$ and $x \in A$, $(a\chi)(x) = \chi(xa) = 0$, i.e., $a\chi = 0$.

THEOREM 19. *Let Q be the canonical two-sided A-module. Then Q is Frobenius. Moreover, for any two-sided ideal $\mathfrak{z}$ of A, its right and left annihilators in Q coincide, and the common annihilator, when regarded as a two-sided $A/\mathfrak{z}$-module, is canonical.*

Proof. We may of course assume that $Q = \bar{A}$. Let $\mathfrak{l}$ be a maximal left ideal of A and suppose that $A - \mathfrak{l} \cong \bar{A}\bar{e}_\kappa$. Then the right annihilator $r_Q(\mathfrak{l})$ is, as right A-module, isomorphic to the right-dual module of $\bar{A}\bar{e}_\kappa$. On the other hand, $r_Q(\mathfrak{l})$ may, since it coincides with the Φ-annihilator of $\mathfrak{l}$ by Lemma 3, be regarded as the Φ-dual module of $A - \mathfrak{l}$, so that we have $r_Q(\mathfrak{l}) \cong \bar{e}_\kappa \bar{A}$ by Lemma 2. The similar holds, by symmetry, for every maximal right ideal of A. In view of the fact that the capacities of both irreducible modules $\bar{A}\bar{e}_\kappa$ and $\bar{e}_\kappa\bar{A}$ coincide with $f(\kappa)$, these, together with the faithfulness of Q (Theorem 18), show that Q is a Frobenius two-sided A-module. Now, Lemma 3 again assures that both the right and the left annihilators $r_Q(\mathfrak{z})$ and $l_Q(\mathfrak{z})$ of a two-sided ideal $\mathfrak{z}$ coincide with the Φ-annihilator of $\mathfrak{z}$ and moreover $\mathfrak{z}$ is the left as well as the right annihilator (in A) of the common annihilator $r_Q(\mathfrak{z}) = l_Q(\mathfrak{z})$. The remaining part of our assertion follows now from Theorem 18, because if μ is a Φ-homomorphism of Q into F as in the theorem then the restriction of μ in the common annihilator satifies the similar conditions as μ does.

We now prove the following fundamental

THEOREM 20. *Let Q be a canonical two-sided A-module. Then, for any finitely generated left A-module M, the right-dual module $M^* = \mathrm{Hom}_A(M, Q)$ of M with respect to Q is, as right A-module, isomorphic to the Φ-dual module $\bar{M}$ of M, by associating each $f \in M^*$ with the composite mapping of f, μ, where μ is a Φ-homomorphism of Q into F as in Theorem* 18.

Proof. For any $f \in M^*$, we denote by $\bar{f}$ the composite mapping of f, μ: $\bar{f}(x) = \mu(xf)$, $x \in M$. Then $\bar{f} \in \bar{M}$, and $(\bar{f}a)(x) = \bar{f}(ax) = \mu(axf) = \mu(xfa)$,

i.e., $\bar{f}a = \widetilde{fa}$ for all $a \in A$, which shows that the mapping $f \to \bar{f}$ gives an A-homomorphism of M^* into $\bar{M}$. Suppose now $\bar{f} = 0$, that is, $\mu(Mf) = 0$. Since Mf is a left A-submodule of Q, it follows that $Mf = 0$, whence $f = 0$. This means that the mapping $f \to \bar{f}$ is one-to-one. Suppose, on the other hand, that $\mu(xM^*) = 0$. Since xM^* is a right A-submodule of Q, it follows also that $xM^* = 0$ (because if $u \in xM^*$ then $\mu(Au) = \mu(uA) = 0$); but since Q is (quasi-)Frobenius by Theorem 19, it follows that $x = 0$ (Theorem 8). This means that the Φ-annihilator of the homomorphic image of M^* (under the mapping $f \to \bar{f}$) is 0, that is, the image fills up $\bar{M}$. Thus the proof is completed.

Now, let Γ denote the center of A. Then Γ is a commutative algebra over Φ, and in particular it satisfies the minimum condition (for ideals). Let C be the canonical Γ-module, that is, the minimal (Γ-)injective extension of the factor module of Γ modulo its radical. Consider, on the other hand, the Φ-dual module $\bar{\Gamma}$ of (the Γ-module) Γ. Then $\bar{\Gamma}$ (is element-wise commutative with Γ and) is, by Theorem 19, a Frobenius Γ-module. Hence, it follows from Theorem 14 that $\bar{\Gamma}$ is isomorphic to C, or what is the same thing, C is a canonical two-sided module of (the algebra) Γ. Then C has, according to Theorem 18, a Φ-homomorphism ν into F such that $\nu(S) \neq 0$ for every non-zero Γ-submodule S of C. Now, looking upon A as an algebra over Γ in the natural way, we consider a canonical two-sided A-module Q. Then Q has, by Theorem 18, a Γ-homomorphism μ into C such that $\mu(au) = \mu(ua)$, $a \in A$, $u \in Q$, and $\mu(L) \neq 0$ for every non-zero left A-submodule L of Q. It is then easy to see that the composite mapping $u \to \nu(\mu(u))$, $u \in Q$, is a Φ-homomorphism of Q into F and fulfills the same conditions as μ, so that Q is, again by Theorem 18, a canonical two-sided module of the algebra A over Φ. We have thus proved the following

THEOREM 21. *The canonical two-sided A-module is uniquely determined, up to isomorphisms, by the ring A, and is independent of the choice of the base ring.*

Let us now call A a *symmetric algebra* if it is canonical when regarded as a two-sided A-module. The following theorem, which follows immediately from Theorem 18, shows that the notion accords with the old one in the case of algebras over a field (cf. Nakayama [11]):

THEOREM 22. *An algebra A over Φ is symmetric if and only if A has a Φ-homomorphism μ into F such that $\mu(ab) = \mu(ba)$ for $a, b \in A$ and $\mu(\mathfrak{l}) \neq 0$ for any non-zero left ideal $\mathfrak{l}$ of A.*

Finally, we shall give characterizations of Frobenius modules and Fro-

benius algebras in terms of the canonical module. For this purpose, we consider a two-sided A-module Q and an automorphism ϕ of A. For any $u \in Q$ and $a \in A$, we define a new product by setting $u*a = ua^{\phi^{-1}}$. Then it is easy to see that Q is converted into a new two-sided A-module under this multiplication, if the left multiplication of elements of A on Q is taken to be the original one. We shall denote this module by (Q, ϕ).

THEOREM 23. *Let Q be the canonical two-sided A-module. Then a two-sided A-module is Frobenius if and only if it is isomorphic to (Q, ϕ) with some automorphism ϕ of A; and, in this case, ϕ is unique up to inner automorphisms.*

Proof. Let Q' be a Frobenius two-sided A-module. In view of the fact that Q is also Frobenius (Theorem 19), Theorem 14 implies that Q' is, as left A-module, isomorphic to Q and moreover A, as right operator-ring, coincides with the endomorphism ring of Q' as well as of Q. Therefore, if $u \rightarrow u'$, $u \in Q$, is an isomorphism of Q onto Q', we can find an automorphism ϕ of A so that $(ua)' = u'a^{\phi}$, or equivalently, $(u*a)'(= (ua^{\phi^{-1}})') = u'a$ for $u \in Q$, $a \in A$, showing that Q' is isomorphic to (Q, ϕ). That conversely every (Q, ϕ) is Frobenius may also be seen quite easily from Theorem 14, while that two modules (Q, ϕ) and (Q, ψ), with automorphisms ϕ, ψ, are isomorphic if and only if $\phi\psi^{-1}$ is an inner automorphism follows immediately from the fact that any isomorphism betwen the above modules can be given by the right multpilication (in the original sense) of a regular element of A.

THEOREM 24. *An algebra A over Φ is Frobenius if and only if A has a Φ-homomorphism μ into F such that $\mu(\mathrm{I}) \neq 0$ for any non-zero left ideal* I. *And, in this case, there exists an automorphism ϕ of A such that $\mu(ab) = \mu(ba^{\phi})$ for a, $b \in A$.* (Cf. [11] and [12, Theorem 1].)

Proof. Since the Φ-dual module $\tilde{A}$ of A is Frobenius (Theorem 19), it is evident from Theorems 14 and 15 that A is Frobenius if and only if A is isomorphic with $\tilde{A}$ as left A-modules. But if we observe the fact that A and $\tilde{A}$ have the same composition length with respect to Φ (by the Corollary of Proposition 2), it is easy to see that the latter condition may be replaced by the weaker condition that $\tilde{A}$ contains an isomorphic image of A, i.e., there exists a $\mu \in \tilde{A}$ such that $a\mu = 0$, $a \in A$, implies $a = 0$, and this proves the first part of our theorem (because $a\mu = 0$ means nothing but $\mu(Aa) = 0$). Now, since $A\mu = \tilde{A}$ and $\tilde{A}$ is faithful, it follows again from the equality of Φ-lengths of A and $\tilde{A}$ that $\mu A = \tilde{A}$ too. Thus, by associating each $a \in A$ with an $a^* \in A$ such that $a^*\mu = \mu a$, we obtain the desired automorphism ϕ: $a \rightarrow a^*$.

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After the submission of this manuscript, the writer has learned of the publication of the following three papers, in which many similarities are found with the present work:

[a] K. Morita, Y. Kawada and H. Tachikawa, "On injective modules," *Mathematische Zeitschrift*, vol. 68 (1957), pp. 217-226.

[b] H. Tachikawa, "Duality theorem of character modules for rings with minimum condition," *Mathematische Zeitschrift*, vol. 68 (1958), pp. 479-487.

[c] K. Morita, "Duality for modules and its applications to the theory of rings with minimum condition," *Science Reports of the Tokyo Kyoiku Daigaku*, vol. 6 (1958), pp. 83-142.

Indeed, most of our principal results are also obtained in these papers; cf. in particular, [c, Theorem 1.1] and [c, Theorem 6.3].

ERRATA.

à l'article: *Sur les revêtements non ramifiés des variétés algébriques* (vol. 79, 1957, pp. 319-330).

par SERGE LANG et JEAN-PIERRE SERRE.

Soit $f: U \to V$ un revêtement d'une variété algébrique V, soit V' une sous-variété irréductible de V, et soient U_i' les composantes de $f^{-1}(V')$. D'après un théorème de Krull, les facteurs séparables $[U_i': V']_s$ des degrés $[U_i': V']$ vérifient l'inégalité:

(1) $$\Sigma[U_i': V']_s \leqq [U: V].$$

Si de plus (1) est une égalité, on a $[U_i': V']_s = [U_i': V']$.

Dans l'article précité, nous avons écrit à la place de (1) la formule incorrecte suivante:

(2) $$\Sigma[U_i': V'] \leqq [U: V].$$

Cette erreur nous a été signalée par M. Greenberg. Elle n'est d'ailleurs d'aucune conséquence pour la suite de l'article: l'inégalité (2) n'intervenait que dans le lemme 1, et peut y être remplacée par (1), à condition de définir les entiers n_i par $n_i = [U_i': V']_s$.

Quant à la formule (2), elle est vraie si V' est *simple* sur V, en vertu de la théorie des intersections (voir Samuel, *Algèbre locale,* p. 32, cor. 2). Elle est par contre inexacte dans le cas général, comme le montre l'exemple suivant:

Soit X une variété normale, définie sur un corps de caractéristique $p > 0$. Soit $U = X^p$ (produit de la variété p fois avec elle-même), et soit $V = X^{(p)}$ (puissance symétrique p-uple de X); la variété V est quotient de U par le groupe symétrique de degré p, ce qui montre que $[U: V] = p!$. Prenons pour V' l'image de la diagonale Δ de X^p; l'image réciproque de V' dans U est Δ, et l'application $\Delta \to V'$ est bijective; toutefois, *ce n'est pas un isomorphisme*; on constate en effet, par application du théorème des fonctions symétriques, que les fonction rationnelles sur V' s'identifient aux puissances p-ièmes des fonctions rationnelles sur Δ. On a donc $[\Delta: V'] = p^{\dim. X}$, et l'inégalité (2) est en défaut si l'on s'arrange pour que $p^{\dim. X} > p!$; l'exemple le plus simple est $p = \dim. X = 2$. On notera que l'on peut même choisir U

non singulière (par contre, on sait que $V = X^{(p)}$ est toujours singulière lorsque $\dim. X \geqq 2$).

Traduit en termes d'algèbre locale, l'exemple précédent fournit deux anneaux locaux normaux A et B, avec B entier et galoisien sur A, tels que, si k_A et k_B désignent leurs corps des restes, on ait:

$$[k_B : k_A] > [B : A].$$

En prenant une infinité de variables on peut même s'arranger pour que $[k_B : k_A] = \infty$, mais les anneaux A et B ne sont alors plus noethériens.

Correction to the paper "On some invariants of cyclotomic fields" by K. Iwasawa, this Journal, vol. 80 (1958), pp. 773-783.

Lemma 2 on p. 779 should be replaced by the following:

Lemma 2. *Suppose that*

$$\sum a_\xi \zeta_{n+1}{}^{\xi} \equiv 0 \bmod. p^s \qquad (\xi \text{ in } U \bmod. U_n)$$

with a_ξ *in* Q_p. *Then* $a_\xi \equiv a_\omega \bmod. p^s$ *whenever* $\xi \equiv \omega \bmod. U_{n-1}$.

Using this lemma, it follows from (8), after a simple computation, that $S_n(\chi) \equiv 0 \bmod. p$ for any character χ of U satisfying $\chi(U_n) = 1$, $\chi(U_{n-1}) \neq 1$, $\chi(\eta) = \eta^{-a}$. Hence (8) implies $\mu > 0$.

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Horizontal fraction signs should be avoided. Instead of them use either solidus signs / or negative exponents.

Neither a solidus nor a negative exponent is needed in the symbols $\frac{1}{2}$, $\frac{1}{\pi}$, $\frac{1}{2\pi}$, $\frac{1}{2\pi i}$, which are available in regular size type.

Binomial coefficients should be denoted by $C_k{}^n$ and not by parentheses. Correspondingly, for symbols of the type of a quadratic residue character the use of some non-vertical arrangement is usually imperative.

For square roots use either the exponent $\frac{1}{2}$ or the sign $\surd$ without the top line, as in $\surd{-1}$ or $\surd(a+b)$.

Replace $e^{(\ )}$ by $\exp(\ )$ if the expression in the parenthesis is complicated.

By an appropriate choice of notations, avoid unnecessary displays.

Simple formulae, such as $A + iB = \frac{1}{2}C^{*}$ or $s_n = a_1 + \cdots + a_n$, should not be displayed (unless they need a formula number).

Use $'$ or $d\ /dx$, possibly D, but preferably not a dot, in order to denote ordinary differentiation and, as far as possible, a subscript in order to denote partial differentiation (when the symbol ∂ cannot be avoided, it should be used as $\partial\ /\partial x$).

Commas between indices are usually superfluous and should be avoided if possible.

In a determinant use a notation which reduces it to the form $\det a_{ik}$.

Subscripts and superscripts cannot be printed in the same vertical column, hence the manuscript should be clear on whether $a_i{}^k$ or $a^k{}_i$ is preferred. (Correspondingly, the limits of summation must not be typed *after* the Σ-sign, unless either $\Sigma_1{}^m$ or $\Sigma^m{}_1$ is desired.) If a letter carrying a subscript has a prime, indicate whether $x_i{}'$ or $x'{}_i$ is desired.

Experience shows that a tilde or anything else *over* a letter is very unsatisfactory. Such symbols often drop out of the type after proof-reading and, when they do not, they usually appear uneven in print. For these reasons we advise against their use. This advice applies also to a bar over a Greek or German letter (for the symbol of complex conjugation an asterisk is often allowed by the context). Type carrying bars over ordinary size italic letters of the Latin alphabet is available.

Bars reaching over several letters should in any case be avoided (in particular, type lim sup and lim inf instead of lim with upper and lower bars).

Repeated subscripts and superscripts should be used only when they cannot be avoided, since the index of the principal index usually appears about as large as the principal index. Bars and other devices *over* indices cannot be supplied. On the other hand, an asterisk or a prime (to be printed *after* the subscript) is possible on a subscript. The same holds true for superscripts.

Distinguish carefully between l. c. "oh," cap. "oh" and zero. One way of distinguishing them is by underlining one or two of them in different colors and explaining the meaning of the colors.

Distinguish between ϵ (epsilon) and $\in$ or ε (symbol), between x (eks) and $\times$ (multiplication sign), between l. c. and cap. phi, between l. c. and cap. psi, between l. c. k and kappa and between "ell" and "one" (for the latter, use ł and 1 respectively).

Avoid unnecessary footnotes. For instance, references can be incorporated into the text (parenthetically, when necessary) by quoting the number in the bibliographic list, which appear at the end of the paper. Thus: "[3], pp. 261-266."

Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Société Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

In any case, the references should be precise, unambiguous and intelligible.

Usually sections numbers and section titles are printed in bold face, the titles "Theorem," "Lemma" and "Corollary" are in caps and small caps, "Proof," "Remark" and "Definition" are in italics. This (or a corresponding preference) should be marked in the manuscript. Use a period, and *not* a colon, after the titles Theorem, Lemma, etc.

German, script and bold face letters should be underlined in various colors and the meaning of the colors should be explained. The same device is needed for Greek letters if there is a chance of ambiguity. In general, mark all cap. Greek letters.

All instructions and explanations for the printer can conveniently be collected on a separate sheet, to be attached to the manuscript.

In case of doubt, recent issues of the Journal may be consulted.